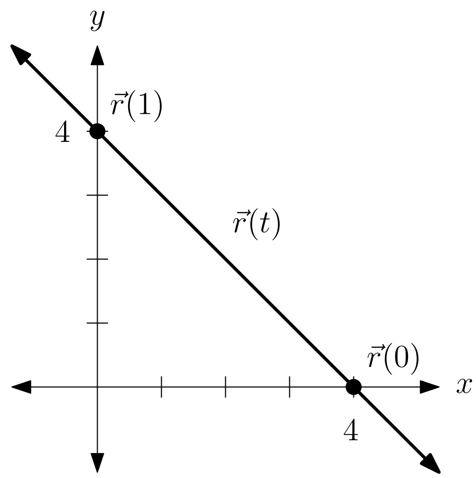
Midterm #1 for MATH-UA.0123-001 [*75 points*]



Problem 1. Let $\vec{r}(t) = \vec{r}_0 + t\vec{u}$, where $\vec{r}_0 = \vec{r}(0)$, as shown in the figure above. [*15 points*]

(a) Write down $\vec{r}(0), \vec{r}(1)$, and $\vec{u}$. [*3 points*]

From the diagram, we can see that $\vec{r}(0) = (4,0)$ and $\vec{r}(1) = (0,4)$. Then, since $\vec{r}(t) = \vec{r}(0) + t\vec{u}$, we have $\vec{r}(1) = \vec{r}(0) + \vec{u}$ so that $\vec{u} = \vec{r}(1) - \vec{r}(0) = (-4,4)$.

(b) Is $\vec{r}(t)$ parametrized by arc length? If it isn't, reparametrize it so that it *is* parametrized by arc length. [*3 points*]

To check if a curve is parametrized by arc length, we need to verify that $\vec{r}'(t) = 1$ for all t. In this case, we can easily see that $\vec{r}'(t) = \vec{u}$. So we just need to check if $|\vec{u}| = 1$. Well, $\vec{u} = \sqrt{32}$, so $\vec{r}(t)$ is **not** parametrized by arc length.

To parametrize $\vec{r}(t)$ by arc length, we could compute the arc length function:

$$s(t) = \int_0^t |\vec{r}'(t)| ds = \int_0^t \sqrt{32} ds = \sqrt{32} t,$$

solve for t, giving $t(s) = s/\sqrt{32}$, and substitute into $\vec{r}(t)$ to get:

$$\vec{r}(s) = \vec{r}(t(s)) = \vec{r}_0 + \vec{u}\frac{s}{\sqrt{32}}.$$

However, it's simpler to notice that if scale $\vec{u}$ by any nonzero constant, we parametrize the same line. So all we need to do is scale $\vec{u}$ so that it has unit magnitude. This gives the same result.

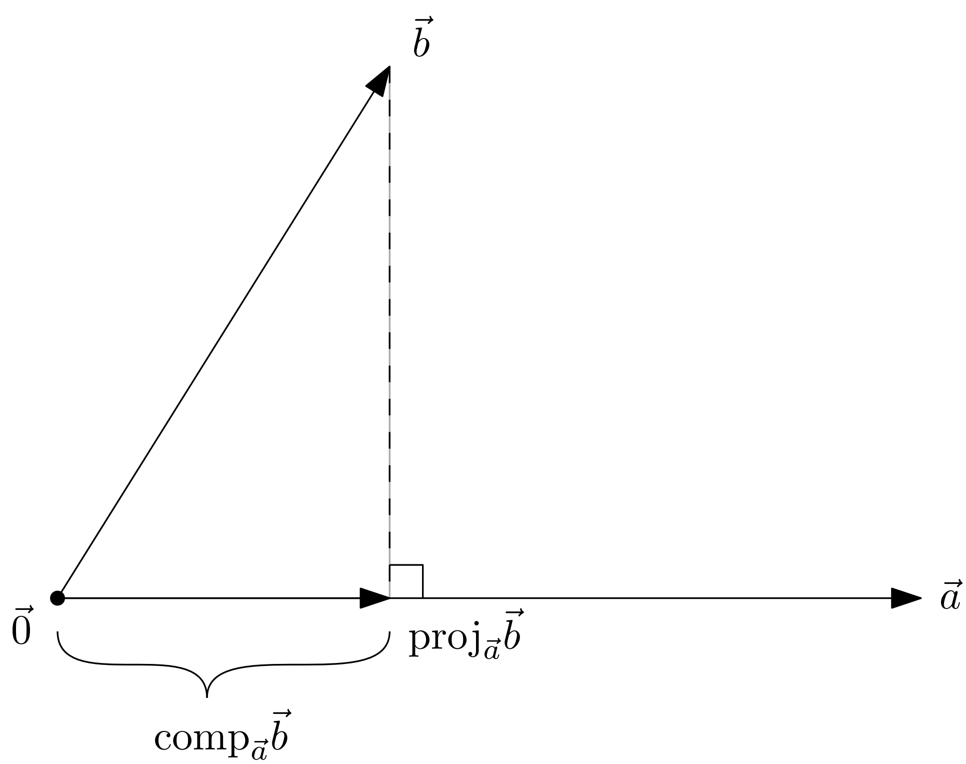
(c) Write the expressions for $\text{comp}_{\vec{a}}\,\vec{b}$ and $\text{proj}_{\vec{a}}\,\vec{b}$, where are $\vec{a}$ and $\vec{b}$ are a pair of vectors. [*3 points*]

The expressions are:

$$\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b}, \qquad \text{and} \qquad \text{proj}_{\vec{a}}\,\vec{b} = \text{comp}_{\vec{a}}\,\vec{b}\frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a}}{|\vec{a}|}\cdot\vec{b}\right)\frac{\vec{a}}{|\vec{a}|}.$$

(d) Draw a picture illustrating $\text{comp}_{\vec{a}}\,\vec{b}$ and $\text{proj}_{\vec{a}}\,\vec{b}$. [*3 points*]

The key point here is that $\text{proj}_{\vec{a}}\,\vec{b}$ is a vector while $\text{comp}_{\vec{a}}\,\vec{b}$ is a scalar:



(e) What point does $P(\vec{x}) = \text{proj}_{\vec{a}}(\vec{x} - \vec{r}_0) + \vec{r}_0$ give? That is, describe in a few words the geometric significance of the function $P(\vec{x})$. [*3 points*]

Let's consider the function $P(\vec{x})$ a little bit at a time. First, $\text{proj}_{\vec{u}}(\vec{x}-\vec{r}_0)$ projects $\vec{x}-\vec{r}_0$ onto $\vec{u}$. However, the second part of the problem indicates that we need to do a little more than just write down some formulas. The hint is in the inclusion of $\vec{r}_0$ and $\vec{u}$ in this problem.

Note that subtracting $\vec{r}_0$ from $\vec{r}(t)$ has the effect of translating the line $\vec{r}(t)$ so that it passes through the origin. Furthermore, if $\vec{y}$ is a point in this new coordinate system, $\text{proj}_{\vec{u}}\,\vec{y}$ has the effect of projecting $\vec{y}$ onto the span of the line $\vec{r}(t)-\vec{r}_0$. Since $\vec{x}-\vec{r}_0$ *is itself* a point in this new coordinate system, we can see that $\text{proj}_{\vec{u}}(\vec{x}-\vec{r}_0)$ is the projection of the point $\vec{x}$ onto the span of the line $\vec{r}(t)$ after the original coordinate system has been shifted by $\vec{r}_0$. Finally, adding $\vec{r}_0$ to $\text{proj}_{\vec{u}}(\vec{x}-\vec{r}_0)$ returns this projected point back to the original coordinate system.

What is the geometric significance of $P(\vec{x})$? Well, what is the geometric significance of $\text{proj}_{\vec{u}}(\vec{y})$? It's the point on the span of $\vec{u}$ which is closest to $\vec{y}$. Hence, $P(\vec{x})$ **is the point on the line $\vec{r}(t)$ which is closest to the point $\vec{x}$.**

Problem 2. Let $\vec{a}=(2,3,4)$, $\vec{b}=(3,3,3)$, and $\vec{c}=(-1,2,2)$. [*11 points*]

(a) Consider the triangle with vertices $\vec{a},\vec{b}$, and $\vec{0}$. What is the area of this triangle? [*3 points*]

Two of this triangles edges correspond to the line segments connecting $\vec{0}$ and $\vec{a}$, and $\vec{0}$ and $\vec{b}$. In class we found that the area of a triangle in $\mathbb{R}^3$ is equal to $|\vec{a}\times\vec{b}|/2$. We have:

$$\vec{a}\times\vec{b}=(-3,6,-3).$$

Then:

$$\frac{|\vec{a}\times\vec{b}|}{2}=\frac{\sqrt{3^2+6^2+3^2}}{2}=\frac{\sqrt{54}}{2}.$$

(b) What are the angles of the triangle in part (a)? You can leave your answer unsimplified in terms of $\cos^{-1}$. [*4 points*]

The edges of the triangle correspond to the vectors $\vec{a}$, $\vec{b}$, and $\vec{b}-\vec{a}$. This is "up to sign"—we can replace any of these vectors with their negations. When we compute the angles, we need to make sure each of the vectors is oriented correctly with respect to the other. The three come from the pairs:

$$\underbrace{\left(\vec{a},\vec{b}\right)}_{\text{Angle \#1}},\qquad \underbrace{\left(\vec{b}-\vec{a},-\vec{a}\right)}_{\text{Angle \#2}},\quad\text{and}\quad \underbrace{\left(\vec{a}-\vec{b},-\vec{b}\right)}_{\text{Angle \#3}}.$$

If you can't picture this and see why this is so, make sure to draw a picture and convince yourself of it.

The angles are then just given by the usual formulas for each of these pairs:

$$\theta_1 = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{27}{\sqrt{29}\sqrt{27}}\right),$$
$$\theta_2 = \cos^{-1}\left(\frac{\left(\vec{b}-\vec{a}\right)\cdot(-\vec{a})}{|\vec{b}-\vec{a}||-\vec{a}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{29}}\right),$$
$$\theta_3 = \cos^{-1}\left(\frac{\left(\vec{a}-\vec{b}\right)\cdot\left(-\vec{b}\right)}{|\vec{a}-\vec{b}||-\vec{b}|}\right) = \cos^{-1}\left(\frac{0}{\sqrt{2}\sqrt{27}}\right).$$

Notice that $\theta_3 = \cos^{-1}(0) = \pi/2$. This is a right triangle! One way we could simplify this problem for ourselves is by first computing the dot products:

$$\vec{a}\cdot\vec{a} = 29, \qquad \vec{b}\cdot\vec{b} = 27, \qquad \left(\vec{b}-\vec{a}\right)\cdot\left(\vec{b}-\vec{a}\right) = 2.$$

We need to compute these anyway, since (e.g.) $\vec{a}\cdot\vec{a} = |\vec{a}|^2$. But the Pythagorean theorem tells us that for a triangle with side lengths A, B, and C, if $A^2 + B^2 = C^2$, then the triangle is a right triangle. In this case:

$$\left|\vec{b}\right|^2 + \left|\vec{b}-\vec{a}\right|^2 = |\vec{a}|^2,$$

from which we can conclude that the angle between $-\vec{b}$ and $\vec{a}-\vec{b}$ is $\pi/2$ (i.e. 90°). From there, we can use SOHCAHTOA to find the other two angles.

(c) Consider the parallelipiped determined by $\vec{a}, \vec{b}$, and $\vec{c}$. What is its volume? [*4 points*]

The volume of a parallelipiped genearted by vectors $\vec{a}, \vec{b}$, and $\vec{b}$ is equal to the absolute value of the scalar triple product of the vectors. Note that because of the absolute value, we can choose any of the six possible "absolute value scalar triple product"s we like (that is, we can put the $\vec{a}$, $\vec{b}$, and $\vec{c}$ wherever we want). This follows from symmetry. Since we likely have already computed $\vec{a}\times\vec{b}$ by this point, it's simplest to compute:

$$\text{volume} = \left|\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)\right| = (-1,2,2)\cdot(-3,6,-3) = 3 + 12 - 6 = 9.$$

Problem 3. Consider the plane which contains the triangle from the previous problem as well as the plane containing the points $\vec{a}, \vec{b}$, and $\vec{c}$. [*5 points*]

(a) What is the angle between these planes? [*4 points*]

The angle between two planes is the same as the angle between their normal vectors. A normal vector for the first plane is:

$$\vec{n}_1 = \vec{a} \times \vec{b} = (-3, 6, -3)\,.$$

A normal for the other plane is:

$$\vec{n}_2 = \left(\vec{b} - \vec{a}\right) \times (\vec{c} - \vec{a}) = (-1, 5, -1)\,.$$

Then, the angle between the two planes is:

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right) = \cos^{-1}\left(\frac{36}{\sqrt{54}\sqrt{27}}\right).$$

Note that there are actually two different angles between the planes, corresponding to whether the argument of $\cos^{-1}$ is positive or negative. However, this opposite angle is just given by $\pi - \theta$.

(b) What is the distance between them? [*1 point*]

In general, to find the distance between two planes, we have two cases: 1) the planes intersect, 2) they do not. If the planes intersect, the distance between them is zero! These planes intersect, since the point $\vec{a}$ is contained in both.

Problem 4. Consider the surface $z = 2x^2 + 3y^2 + 4x - 3y + 6$. What kind of quadric is this? [*5 points*]

As it's written, $z = z(x, y)$ isn't written in one of the standard forms for a quadric. To get it into the right form, we complete the square:

$$z = 2(x+1)^2 + 3(y - \tfrac{1}{2})^2 + \frac{15}{4}.$$

So, we can see that this is an elliptic paraboloid.

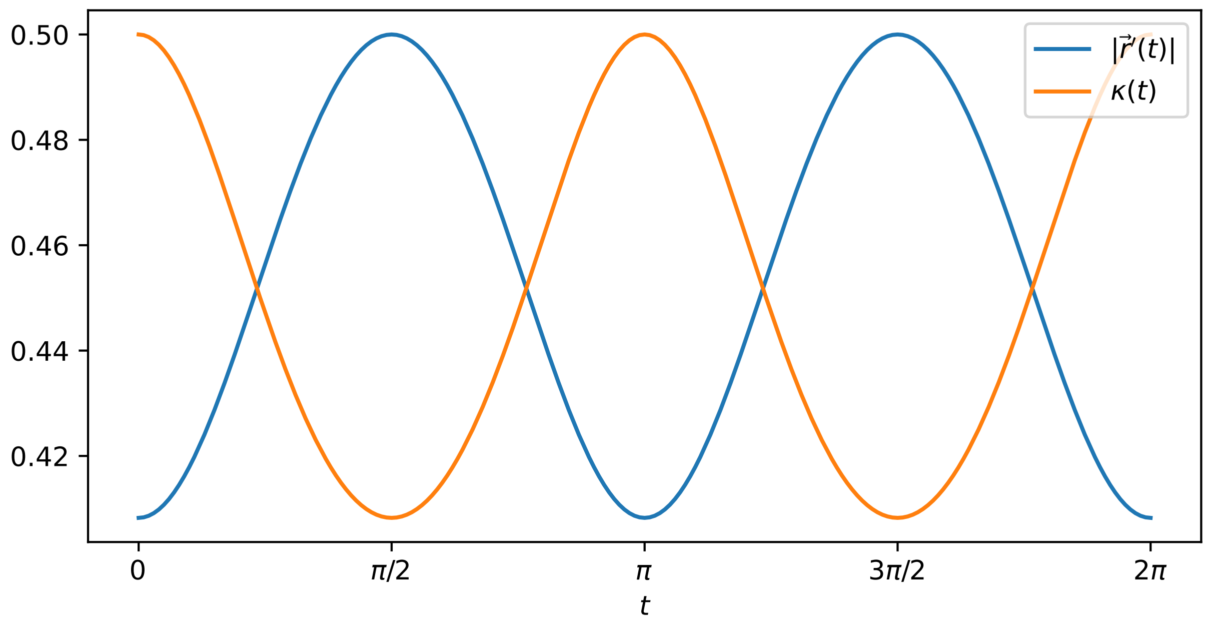


Figure 1: Plots of $|r'(t)|$ and $\kappa(t)$ for t such that $0 \le t \le 2\pi$.

Problem 5. Let $\vec{r}(t) = \left(\frac{1}{2}\cos(t) - 2, \frac{1}{\sqrt{6}}\sin(t) + 1, -\frac{1}{2}\right)$. [*12 points*]

(a) Write an integral expression for $s(t)$ starting from $t = 0$. [*3 points*]

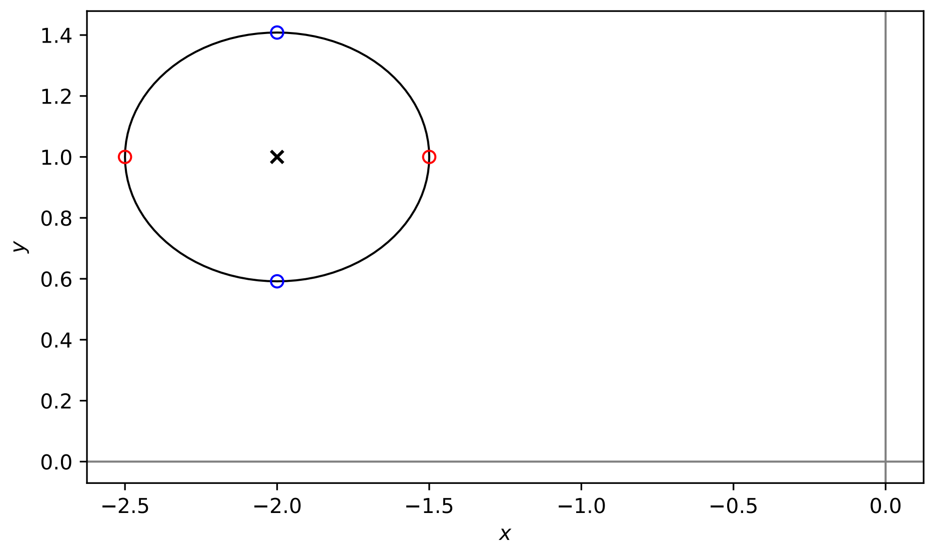
The arc length function is:

$$s(t) = \int_0^t \left|\frac{d\vec{r}}{dt}\right| d\sigma = \int_0^t \sqrt{\frac{\sin(\sigma)^2}{4} + \frac{\cos(\sigma)^2}{6}} d\sigma.$$

If you try to explicitly evaluate this integral, you'll find that you can't. That's why we leave it as it is in this integral form.

(b) Sketch the trajectory of $\vec{r}(t)$ in the xy-plane. Where is $|d\vec{r}/dt|$ maximimized and minimized on this trajectory? [*3 points*]

A sketch of the curve $\vec{r}(t)$ looks like this in the xy-plane (for $z = -1/2$):



From the previous problem, we had:

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{\sin(t)^2}{4} + \frac{\cos(t)^2}{6}}.$$

We can rewrite this as:

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{6\sin(t)^2 + 4\cos(t)^2}{24}} = \sqrt{\frac{4 + 2\sin(t)^2}{24}} = \sqrt{\frac{1}{6} + \frac{\sin(t)^2}{12}}.$$

We can see that this function is maximized when $\sin(t)^2$ is maximized, and minimized when it's minimized. So, $|d\vec{r}/dt|$ has a local minimum when $t = \pi k$ for $k \in \mathbb{Z}$, and a maximum when $t = \pi k + \frac{\pi}{2}$ for $k \in \mathbb{Z}$.

An exercise for you: For this problem, since we're familiar with the square root function and $\sin(x)$, we can tell pretty easily what's going on. However, if you like, try taking the first and second derivatives of $|d\vec{r}/dt|$ as a function of t and putting the previous argument on surer footing.

(c) Compute $\kappa(t)$. Where is $\kappa(t)$ maximized and minimized on $\vec{r}(t)$? [*6 points*]

Problem 6. Let $f(x, y, z, t) = F(ax + by + cz - t)$, where $a, b, c \in \mathbb{R}$. What conditions must a, b, and c satisfy in order for $f_x + f_y + f_z + f_t = 0$ to hold? [*5 points*]

Problem 7. Consider the equation $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$. [*12 points*]

(a) Solve for $z = z(x, y)$. What is the domain of z? [*3 points*]

(b) Sketch the $z = 0, 1, 2, 3$, and 4 level sets. [*5 points*]

(c) Form the tangent plane approximation to z at $(1, 0)$. Approximate $z(0, 0)$ using this linearization. What is the error? [*4 points*]

Problem 8. Let $w = f(x, y, z)$, $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$. [*10 points*]

(a) Write the expressions for $\partial w/\partial u$ and $\partial w/\partial v$ using the chain rule. [*4 points*]

(b) Let $w = 3x^2 - 2xy + 4z^2$, $x = e^u \cos v$, $y = e^u \sin v$, and $z = e^u$. Write down explicit expressions for $\partial w/\partial u$ and $\partial w/\partial v$. [*6 points*]