Midterm #2 for MATH-UA.0123-001

Problem 1. Let $f(x,y) = \cos(x^2+2y)$, and let $\boldsymbol{u} = (\cos(\theta), \sin(\theta))$. Compute $D_{\boldsymbol{u}} f(\sqrt{\frac{\pi}{2}}, -\frac{\pi}{2})$ for $\theta = \pi/4$.

The directional derivative $D_{\boldsymbol{u}} f$ equals $\boldsymbol{u} \cdot \nabla f$. The gradient of f is:

$$\nabla f(x,y) = \Big(-2x\sin(x^2+2y), -2\sin(x^2+2y)\Big).$$

Note that for $x = \sqrt{\pi/2}$ and $y = -\pi/2$, we have $\sin(x^2+2y) = \sin(\pi/2 - \pi) = \sin(-\pi/2) = -1$, so that $\nabla f(\sqrt{\pi/2}, -\pi/2) = 2(\sqrt{\pi/2}, 1)$. Then, for $\theta = \pi/4$:

$$D_{\boldsymbol{u}} \nabla f(\sqrt{\tfrac{\pi}{2}}, -\tfrac{\pi}{2}) = 2\Big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\Big) \cdot \Big(\sqrt{\frac{\pi}{2}}, 1\Big) = \sqrt{\pi} + \sqrt{2}.$$

Problem 2. Consider the sphere of radius r whose center is the origin. Show that the normal line for each point on the sphere passes through the origin. *Hint*: start by writing down a level set function $f(x,y,z)$ such that $f(x,y,z) = 0$ is the sphere of interest.

In general, the normal line of a surface at a point $\boldsymbol{x}_0$ is any point which lies on the line $\boldsymbol{r}(t) = \boldsymbol{x}_0 + t\boldsymbol{n}(\boldsymbol{x}_0)$, where $\boldsymbol{n}$ is the surface normal of that surface. If we can describe a surface using a level set function, say $f(\boldsymbol{x}) = 0$, then the normal of that surface at a point $\boldsymbol{x}$ on the surface is $\nabla f(\boldsymbol{x})$.

We can write the sphere of radius r in $\mathbb{R}^3$ using a level set function as:

$$f(x,y,z) = x^2 + y^2 + z^2 - r^2 = 0.$$

The sphere's surface normal is:

$$\nabla f(x,y,z) = \Big(2x, 2y, 2z\Big).$$

Let (x,y,z) be a point on the sphere. Then, we can parametrize that normal line through (x,y,z) as:

$$\boldsymbol{r}(t) = (x,y,z) + t\Big(2x, 2y, 2z\Big) = \Big((2t+1)x, (2t+1)y, (2t+1)z\Big).$$

Then:

$$\boldsymbol{r}(-1/2) = (0,0,0).$$

Hence, the normal line for each point on the sphere passes through the origin.

Problem 3. Use the method of Lagrange multipliers to prove that the rectangle of maximum area with a given perimeter p is a square.

Let the width of a rectangle be w, and its height be h. Then its perimeter is $p(w,h) = 2w+2h$, and its area is $A(w,h) = wh$. So, the maximization problem we're interested in solving is:

$$\begin{array}{ll} \text{maximize} & wh \\ \text{subject to} & 2w + 2h \end{array}$$

To use the method of Lagrange multipliers, we need to find a Lagrange multiplier λ such that $\nabla A(w,h) = \lambda \nabla p(w,h)$. This is equivalent to solving the system:

$$h = 2\lambda, \qquad w = 2\lambda$$

for w, h, and λ. We conclude that $\lambda^* = w/2 = h/2$. Hence, $w^* = h^*$.

Note that this just gives us a first-order necessary condition for optimality. That is, we know that $w^* = h^*$ such that $p = 2w^* + 2h^*$ is a critical point, but we don't know if it's a local maximum or a local minimum. In terms of how this problem was graded, that was good enough for me. Another way to do this problem is as follows.

Since p is constant with respect to w and h, we can eliminate the equality constraint from the problem by writing $h(w) = p/2 - w$, so that $A(w) = w(p/2 - w) = pw/2 - w^2$. Then, $A'(w) = p/2 - 2w$. If $A'(w) = 0$, then $w = p/4$, which gives $h(w) = h(p/4) = p/2 - p/4 = p/4$. So, we again see that the width and height must be equal. Now, since $A''(w) = -2 < 0$, we can see that this critical point must be a local maximum (in fact, a global maximum).

Problem 4. Consider the rectangle $R = [0, \pi] \times [0, \pi]$, and the integral:

$$I = \iint_R \cos(x)\sin(y) dA.$$

Use the midpoint rule with $m = n = 2$ (that is, divide R into *four* equal squares *total*) to approximate I. Next, evaluate I by doing the double integral. What is the error in the midpoint rule approximation?

If we divide R into four squares of equal size, their midpoints are:

$$\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}\right), \quad \left(\tfrac{3}{4}\pi, \tfrac{\pi}{4}\right), \quad \left(\tfrac{\pi}{4}, \tfrac{3}{4}\pi\right), \quad \left(\tfrac{3}{4}\pi, \tfrac{3}{4}\pi\right),$$

and their areas are $A = (\pi/2)^2 = \pi^2/4$. Applying the midpoint rule to approximate I gives:

$$\begin{aligned} I_{\text{mp}} &= A \cdot \left(\cos\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{3}{4}\pi\right)\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{3}{4}\pi\right) + \cos\left(\tfrac{3}{4}\pi\right)\sin\left(\tfrac{3}{4}\pi\right)\right)\right) \\ &= A \cdot \left(\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\right) = 0. \end{aligned}$$

On the other hand, if we do the double integral, we get:

$$\begin{aligned} I &= \iint_R \cos(x)\sin(y)dA = \int_0^\pi \cos(x)dx \int_0^\pi \sin(y)dy \\ &= \sin(x)|_{x=0}^{\pi} - \cos(y)|_{y=0}^{\pi} = (\sin(\pi) - \sin(0))(\cos(0) - \cos(\pi)) \\ &= (0-0)(1-1) = 0. \end{aligned}$$

Since $I_{\text{mp}} = I$, there is no error in the midpoint rule approximation.

Problem 5. Evaluate $\iiint_E \sqrt{x^2+y^2}$ where E is the region that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$, and between the planes $z = z_0$ and $z = z_1$, where $z_0 < z_1$.

Let $\Delta z = z_1 - z_0$. Then:

$$\iiint_E \sqrt{x^2+y^2} = \int_0^{2\pi}\int_{z_0}^{z_1}\int_2^4 r^2 dr dz d\theta = 2\pi\Delta z \left.\frac{r^3}{3}\right|_{r=2}^{4} = \frac{2}{3}\pi\Delta z\,(64-8) = \frac{112}{3}\pi\Delta z.$$

Problem 6. Set up and evaluate a triple integral in spherical coordinates to find the volume of a sphere of radius r.

$$\begin{aligned} \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^2\sin(\phi)d\rho d\phi d\theta &= 2\pi\int_0^\pi \sin(\phi)d\phi \int_0^\rho \rho^2 d\rho \\ &= 2\pi\cdot\Big(-\cos(\phi)|_{\phi=0}^{\pi}\Big)\cdot\left(\left.\frac{\rho^3}{3}\right|_{\rho=0}^{r}\right) = \frac{4}{3}\pi r^3. \end{aligned}$$

Problem 7. Find the average distance between the origin and a point in a spherical shell (centered about the origin) with inner and outer radii $r_0 < r_1$.

The average value of a function f over a three-dimensional region R is given by:

$$\frac{1}{\text{vol}(R)}\iiint_R f(x,y,z)dA.$$

From the previous problem, we know that the volume of a sphere of radius r is $4\pi r^3/3$. So, a spherical shell with inner and outer radii r_0 and r_1 has volume given by $4\pi(r_1^3 - r_0^3)/3$. The function we want to average is just the Euclidean distance of a point to the origin, which is ρ in spherical coordinates. So, we start by integrating:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_{r_0}^{r_1}\rho^3\sin(\phi)d\rho d\phi d\theta &= 2\pi\int_0^{\pi}\sin(\phi)d\phi\int_{r_0}^{r_1}\rho^3 d\rho\\ &= 4\pi\cdot\left.\frac{\rho^4}{4}\right|_{\rho=r_0}^{r_1} = 4\pi\cdot\frac{r_1^4-r_0^4}{4} = \pi(r_1^4-r_0^4).\end{aligned}$$

Hence, the average distance over the shell is:

$$\frac{\pi(r_1^4-r_0^4)}{\frac{4}{3}\pi\left(r_1^3-r_0^3\right)} = \frac{3}{4}\cdot\frac{r_1^4-r_0^4}{r_1^3-r_0^3}.$$

Problem 8 (*bonus*). Observe that if $r_0 = r_1$ in Problem 7, the average distance is obviously just r_0, since the average distance to the origin of a point on a sphere is just the radius of the sphere itself. Prove this by taking the limit as $r_1 \to r_0$ of the result of Problem 7.

The issue with our formula for the average distance is that it is undefined if we set $r_0 = r_1$. If we take the limit as $r_1 \to r_0$, it is in indeterminate form. To get around this, we could use L'Hôpital's rule.

$$\frac{3}{4}\lim_{r_1\to 0}\frac{r_1^4-r_0^4}{r_1^3-r_0^3} = \frac{3}{4}\lim_{r_1\to r_0}\frac{4r_1^3}{3r_1^2} = r_0.$$

Another approach is to factor the numerator and denominator of our expression. The numerator can be factored as:

$$r_1^4-r_0^4 = \left(r_1^2\right)^2-\left(r_0^2\right)^2 = \left(r_1^2-r_0^2\right)\left(r_1^2+r_0^2\right) = (r_1-r_0)(r_1+r_0)(r_1^2+r_0^2),$$

and the denominator can be factored as:

$$r_1^3-r_0^3 = (r_1-r_0)(r_1^2+r_0r_1+r_0^2).$$

Hence, if we set $r = r_1 = r_0$, we get:

$$\frac{3}{4}\frac{(r+r)(r^2+r^2)}{r^2+r\cdot r+r^2} = \frac{3}{4}\frac{2r\cdot 2r^2}{3r^2} = \frac{3}{4}\cdot\frac{4r^3}{3r^2} = r.$$

Factoring the numerator and denominator and cancelling one of the factors shows that the expression has a "removable singularity". That is, the formula for the average distance of a point on a spherical shell to the origin isn't "truly" singular, which matches our physical intuition.

Problem 9. Find the volume of the solid E that lies below the cone $z = -\sqrt{x^2+y^2}$ and above the sphere $x^2+y^2+z^2=r^2$.

By symmetry, this is the same as the volume of the solid which lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = r^2$ (we can obtain this shape by reflecting the original shape over the xy plane, which doesn't change its volume). We computed this volume in class:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/4}\int_0^r \rho^2\sin(\phi)d\rho d\phi d\theta &= 2\pi\int_0^{\pi/4}\sin(\phi)d\phi\int_0^r \rho^2 d\rho \\ &= 2\pi\cdot(\cos(0)-\cos(\pi/4))\cdot\frac{r^3}{3} = \frac{2}{3}\pi\left(1-\frac{\sqrt{2}}{2}\right)r^3.\end{aligned}$$

Problem 10. Consider the vector field $\boldsymbol{F}(x,y) = (-y,x)$. Is this a gradient vector field? Why or why not? Let C be a circular arc of radius r subtending an angle θ. Evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$.

If we integrate the first component of $\boldsymbol{F}$ with respect to x, we get $-xy+C$, for some constant C. On the other hand, if we integrate the second component with respect to y, we get $xy+C'$, for another constant C'. So, we can see that $\boldsymbol{F}$ isn't a gradient vector field.

Since $\boldsymbol{F}$ isn't conservative, to compute the line integral, we have to integrate it directly. We can parametrize a circular arc of angle θ and radius r as:

$$\boldsymbol{r}(t) = r(\cos t,\sin t), \qquad 0\le t\le\theta.$$

Then, $\boldsymbol{r}'(t) = r(-\sin t,\cos t)$. Hence:

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^\theta \boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt \\ &= \int_0^\theta r(-\sin t,\cos t)\cdot r(-r\sin t,\cos t)dt \\ &= r^2\int_0^\theta\left(\sin(t)^2+\cos(t)^2\right)dt = r^2\int_0^\theta dt = \theta r^2.\end{aligned}$$

Problem 11. Let $\boldsymbol{F}(x,y) = (xy^2, x^2y)$, and let C be the unit circle. What is $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$?

Let $f(x,y) = x^2y^2/2$. Then, $\boldsymbol{F} = \nabla f(x,y)$, from which we see that $\boldsymbol{F}$ is a conservative vector field on all of $\mathbb{R}^2$. Since C is a closed loop, we can conclude that $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$.

Problem 12 (*bonus*). A *spherical rectangle* is a set $[\rho_0,\rho_1]\times[\theta_0,\theta_1]\times[\phi_0,\phi_1]\subseteq\mathbb{R}^3$, parametrized using spherical coordinates. Assuming that $\rho_0\le\rho_1, \theta_0\le\theta_1$, and $\phi_0\le\phi_1$, draw or name as many different kinds of shapes that you can think of which are actually just spherical rectangles. (If your drawings suck, you won't get any points.)