Midterm #2 Study Guide

11.6: Directional derivatives and the gradient

In 2D and 3D, understand:

- the limit definition of the directional derivative
- how to compute a directional derivative using Theorem 3
- the definition of the gradient
- the fact that the directional derivative is just the dot product between $\vec{u}$ and the ∇f

Also, understand:

- that the gradient $\nabla f(x)$ is a vector which points in the direction of steepest ascent in f at the point x (Theorem 15)
- the difference between the direction of steepest increase and the rate of steepest increase (Example 6)
- that the gradient of a level set function is a normal to the tangent plane of the level set at that point (Equation 19)
- that the gradient of that same level set function spans the normal line of the level set at that point

See Example 7 regarding the last two points.

Practice problems: 20, 21, 25, 29, 39, 51.

11.7: Maximum and minimum values

Understand:

- the definition of local maxima and minima
- the definition of absolute maxima and minima
- the first-order necessary condition for optimality (Theorem 2)
- the definition of a critical point
- the conditions under which a critical point is a local max or min
- how to find the critical points of a function

- the second-sufficient condition for optimality of a function in two variables (the "2D second derivative test")—see Theorem 3 and Example 3
- how to model and solve simple geometry problems by minimizing or maximizing functions—see Example 4 and Example 5

Practice problems: 28, 31, 39, 45.

Additional practice problem: Find the critical points of $f(x, y) = \cos(x + y)\sin(x - y)$. Hint: there are infinitely many. Double hint: if you can, always make a plot. For more practice, try making a plot by hand before using a computer.

11.8: Lagrange multipliers

Understand:

- the definition of a Lagrange multiplier
- the method of Lagrange multipliers
- the kind of minimization and maximization problems the method of Lagrange multipliers is useful for
- ...and the kind of problems it is NOT useful for
- **very important**: the geometric significance of Lagrange multipliers, i.e. be able to draw a picture and explain why Equation 1 on page 678 must be true
- how to apply the method of LMs to problems with more than one constraint
- that the number of equations in the method of Lagrange multipliers must equal "the number of variables" + "the number of Lagrange multipliers"

The examples in this chapter are good practice.

Practice problems: 25, 28, 29, 37, 43.

12.1: Double integrals over rectangles

Understand:

- what something like $[a, b] \times [c, d]$ means
- how to compute the volume of a solid that lies above a rectangle

- what a double Riemann sum is, and have an idea of how to use it to estimate a double integral over a rectangle—mainly, understand the midpoint rule
- what an iterated integral is and how Equation 7 differs from the left-hand side of Equation 6
- what Fubini's theorem is, and what it lets us do—i.e., change the order of integration under certain conditions on the integrand (**there will be no trick questions where Fubini's theorem doesn't apply on this test**)
- how to use Fubini's theorem to make iterated integrals easier to solve
- how to use Equation 11 to make your life easier
- the properties double integrals
- how to use Inequality 14 to get lower and upper bounds on the volume of a region

Practice problems: 4, 7–9, 11–26, 34, 41, 42.

12.2: Double integrals over general regions

Understand:

- Type I and Type II regions
- how to draw these regions and understand what is happening when we do an iterated integral (first we integrate over one direction, and then for each point, we integrate along the other direction)
- understand why we can no longer simply change the order of integration, and why we must determine new limits of integration in order to do so
- **know how to do this by making a picture!** (see Figures 9, 10, and 12)

Practice problems: 13–14, 43–48 (draw a picture for these!), 50.

12.3: Double integrals in polar coordinates

Understand:

- how to convert back and forth between polar coordinates and Cartesian coordinates
- what a polar rectangle is
- Equation 2—**DON'T FORGET THE EXTRA FACTOR OF** r

- how to parametrize different kinds of regions using polar coordinates
- when a region will be easier to parametrize using polar coordinates than Cartesian coordinates, and vice versa

Practice problems: 13–19, 23–26, 29.

12.5: Triple integrals

This section is like 12.1 combined with 12.2 but in 3D. The same considerations apply. The key difference is that the integrals are now somewhat more complicated, and there is more bookkeeping that can trip you up. The hardest part of this chapter is visualizing 3D volumes, and figuring out how to change the order of integration of general regions.

Practice problems: 7–20, 31, 32. If you really want to punish yourself, do 27–30 and 33–34.

12.6: Triple integrals in cylindrical coordinates

This is a short chapter. Understand:

- cylindrical coordinates
- how to convert back and forth between cylindrical and Cartesian coordinates
- how to use Equation 4—**DON'T FORGET THE EXTRA FACTOR OF** r

You should also understand what a "rectangle" (i.e., $[r_0, r_1] \times [\theta_0, \theta_1] \times [z_0, z_1]$) in cylindrical coordinates looks like, and how to express common 3D shapes in cylindrical coordinates.

Practice problems: 17–28. Make 3D plots of these volumes by hand if you have time.

12.7: Triple integrals in spherical coordinates

This is also a short chapter, but more challenging than 12.6. Understand:

- spherical coordinates
- how to convert back and forth between spherical and Cartesian coordinates
- how to use Equation 3—**DON'T FORGET THE EXTRA FACTOR OF** $\rho^2 \sin\phi$

Example 4 is worth taking a close look at.

Practice problems: 21–32. Make 3D plots by hand to practice visualizing in 3D. The more you practice making these drawings, the easier it will get to visualize regions accurately without having to burn precious time making a drawing.

13.1: Vector fields

Vector fields are just functions whose input is a point and whose output is a vector. Each of their components are scalar fields. Know how to plot a vector field, and know what a gradient/conservative vector field is, and know what a potential function is. Understand the examples related to gravitation and electric charge.

Practice problems: Try some of 1–10 to practice drawing vector fields. Try 15–18.

13.2: Line integrals

Understand:

- **Equation 3**, and in particular how the choice of parametrization affects the integral on the righthand side of Equation 3 (what happens if the curve is parametrized by arc length?)
- how to integrate a piecwise smooth curve (see page 763)
- Equation 8
- how the orientation of a parametrization affects the sign of a line integral (see the few paragraphs before the start of "Line integrals in Space" on page 766)
- understand that we can write the integral in Equation 3 as:

$$\int_C f(\boldsymbol{x})ds = \int_{t_0}^{t_1} f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|dt,$$

  hence, it makes sense in any number of dimensions, not just two
- understand how to take a line integral in a vector field (in particular, Definition 13—the expression in the middle is the one you will actually use to do integrals, the one on the left is just notation for the expression in the middle, and you are unlikely to encounter a curve parametrized by arc length, so the one of the right is of limited utility)

Practice problems: Try some of 1–16 to practice computing line integrals of scalar fields and 19–22 to practice computing line integrals of vector fields. Try 37–42 for physics examples. Try 45–47 for slightly more "theoretical" problems (these are fairly basic, but are worth thinking about to develop a deeper understanding of how these integrals work beyond just manipulating expressions on paper).

13.3: The fundamental theorem for line integrals

The main thing to understand is Theorem 2. I highly recommend studying its proof. It is simple and instructive. However, the book goes in the wrong direction. It is better to start from the bottom, apply the fundamental theorem of calculus, and compute the derivative using the chain rule. Beyond this, understand:

- The section on "Independence of path", including Theorems 3, 4, 5, and 6
- In particular, understand how to use Theorem 5/6 to determine when a vector field is conservative in 2D (hence, when path independence holds)

As far as a recipe for this chapter goes, the approach to take for solving doing line integrals of vector fields is:

- Determine whether the vector field is conservative, i.e. gradient (in 2D, using Theorem 5/6; in 3D, you will need to find the potential function—i.e., the function whose gradient is the vector field)
- If it is, apply Theorem 2 and compute the integral by taking the difference of the potential at the endpoints—or, if the path is a loop, conclude that the integral is zero. Note that if you used Theorem 5/6, you will still need to find the potential in order to evaluate the integral!
- If it isn't, use the methods of Section 13.2 to solve the line integral over the path of interest (you MUST use the given path, since path independence no longer holds)

Practice problems: 3–16, 31.