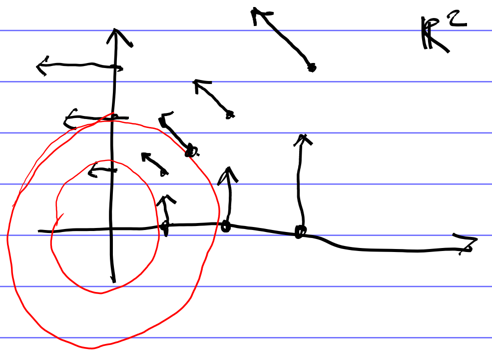
13.1 Vector fields

$\vec{F} : \mathbb{R}^{2,3} \to \mathbb{R}^{2,3}$



$\vec{F}(x,y) = -y\hat{i} + x\hat{j}$

$\vec{r}(t) = (a\cos(t), a\sin(t))$

$\vec{r}\,'(t) = (-a\sin(t), a\cos(t))$

- Examples 4 and 5 in the book ← check these out
- Gradient vector fields, or "conservative" vec. fields.

$\vec{F}(x,y,z) = \nabla f(x,y,z)$

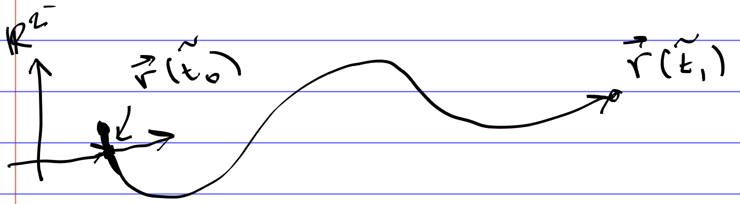
$\Rightarrow \vec{F}$ is gradient / conservative

- important in physics
- important for line integrals

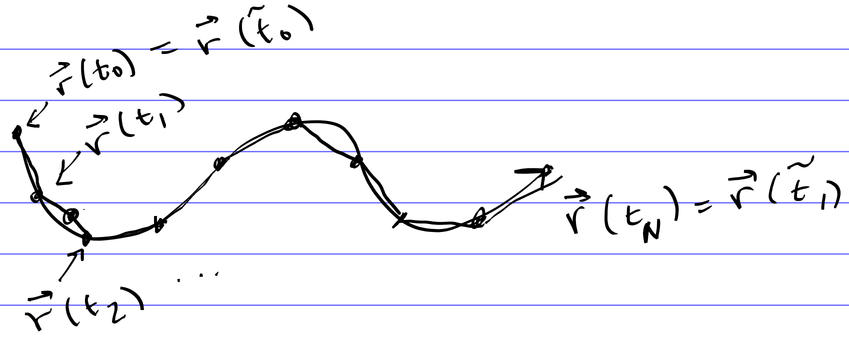
13.2 Line integrals

1) define a Riemann sum approximating a quantity of interest
2) take the limit
3) get an integral

Goal: integrate a scalar field over a curve

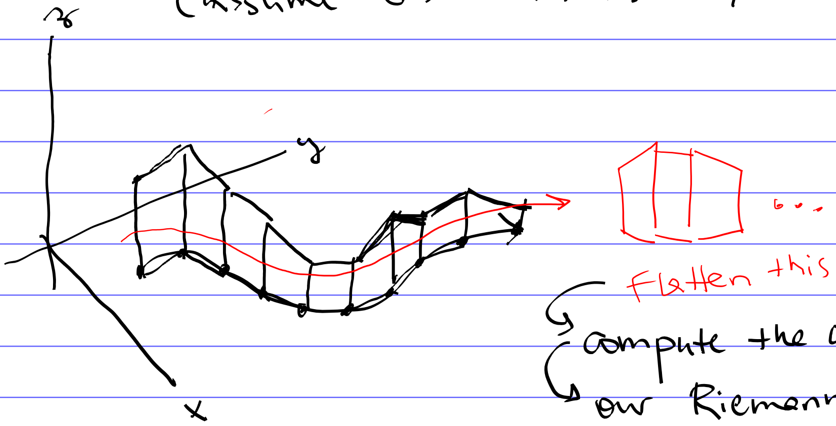


$\vec{r}(t)$, $\tilde{t}_0 \le t \le \tilde{t}_1$, $f : \mathbb{R}^2 \to \mathbb{R}$



$$\sum_{i=1}^{N} f(\vec{r}(t_i)) \|\vec{r}(t_i) - \vec{r}(t_{i-1})\|$$

(assume $f \geq 0$ for a sec...)



Flatten this strip

compute the area

our Riemann sum

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(\vec{r}(t_i)) \|\vec{r}(t_i) - \vec{r}(t_{i-1})\|$$

$$= \cancel{\int_{t_0}^{t_1} f(\vec{r})} \underbrace{ds}_{\text{infinitesimal line element}}$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} f(\vec{r}(t_i)) \underbrace{\left\| \frac{\vec{r}(t_i) - \vec{r}(t_i - 1)}{t_i - t_{i-1}} \right\|}_{\to \vec{r}\,'(t_i) \text{ as } N\to\infty} (t_i - t_{i-1})$$

$$= \int_{t_0}^{t_1} f(\vec{r}(t)) \underbrace{\|\vec{r}\,'(t)\| dt}_{ds}$$

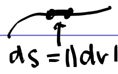
$s_i - s_{i-1}$

δs

ds = infinitesimal line element

$\approx$ a small piece of arc length

$\|dr\| = \|r_i - r_{i-1}\|$ =



$$ds = \|d\vec{r}\| = \left\| \frac{d\vec{r}}{dt} \cdot dt \right\|$$
$$= \left\| \frac{d\vec{r}}{dt} \right\| dt = \|\vec{r}\,'(t)\| dt$$

Line integral: $$\int_{t_0}^{t_1} \underbrace{f(\vec{r}(t))}_{\text{our integrand}} \underbrace{\|\vec{r}\,'(t)\|}_{\text{weight}} dt$$

If $\vec{r}$ is parametrized by arc length, then:

$$\left\| \frac{\vec{r}\,'(s)}{ds} \right\| \equiv 1 \quad \forall s$$

$$\Rightarrow \int_{s_0}^{s_1} f(\vec{r}(s))\, ds$$

<u>Line integrals of vector fields</u> :

$f(x)$: a force (scalar → scalar)

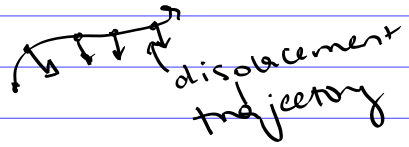
$$\int_a^b f(x)\,dx = W \quad : \text{work } [N \cdot m]$$

OTOH:

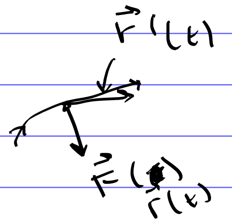
constant force $\vec{F}$ $[N]$, displacement $\overrightarrow{PQ}$ $[m]$

work is: $\vec{F} \cdot \overrightarrow{PQ}$ $[Nm]$

$\vec{F}(\vec{x})$



how do we compute the work?



$$\underbrace{\vec{F}(t)}_{N} \cdot \underbrace{(\vec{r}(t_i) - \vec{r}(t_{i-1}))}_{m}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \approx \underbrace{\vec{F}(\vec{r}(t))}_{N} \cdot \underbrace{\frac{\vec{r}(t_i) - \vec{r}(t_{i-1})}{t_i - t_{i-1}}}_{\frac{m}{s}}$$

$$\vec{F}(\vec{r}(t)) \cdot \underbrace{\vec{r}\,'(t)\, dt}_{d\vec{r}} \quad [N \cdot m]$$

$$\int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_{t_0}^{t_1} \vec{F} \cdot d\vec{r}$$

$$\|$$

$$\underbrace{\int_{t_0}^{t_1} \left[\underbrace{\vec{F}(\vec{r}(t)) \cdot \vec{T}(t)}_{\text{comp}_{\vec{T}(t)} \vec{F}(\vec{r}(t))\ [N]}\right] \underbrace{\|\vec{r}\,'(t)\|}_{[\frac{m}{s}]}\, \underbrace{dt}_{[s]}}_{\text{line integral}}$$

$$\underbrace{[N]\ [\tfrac{m}{s}]\ [s]}_{[Nm]}$$

13.3 The fundamental theorem for line integrals

1) Assume we have a gradient vector field:
$$\vec{F}(\vec{x}) = \nabla f(\vec{x})$$

2) Consider a curve $\vec{r}(t)$

3) Evaluate the difference $f(\vec{r}(t_1)) - f(\vec{r}(t_0))$

$$f(\vec{r}(t_1)) - f(\vec{r}(t_0)) \overset{\text{FTC}}{=} \int_{t_0}^{t_1} \frac{d}{dt}\left\{ f(\vec{r}(t)) \right\} dt \qquad f'(g(x))g'(x)$$

$$= \int_{t_0}^{t_1} \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt = \int_{t_0}^{t_1} \nabla f \cdot d\vec{r}$$

If $\vec{F}$ is conservative, and C is a curve (diff.)

$$\int_C \vec{F} \cdot dr = f_1 - f_0$$

where f_0 and f_1 are the potential generating $\vec{F}$ (i.e. $\vec{F} = \nabla f$) at the endpoints.

this assumes a parametrization of C going from C_0 to C_1 (the endpoints)

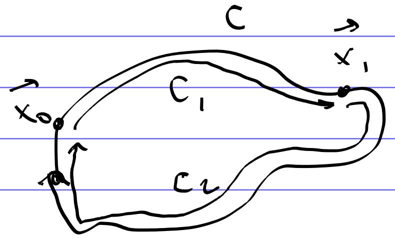
Note: if you reverse the curve, then the integral is negated.

$$\int_{t_0}^{t_1} \nabla f(\vec r(t))\cdot \frac{d\vec r}{dt}\,dt \xrightarrow[t\mapsto t_1 - t]{dt\mapsto -dt} \text{negated}$$

<u>Consequences</u>: $\vec F$ is conservative, and C is a closed loop,

then: $\int_C \vec F\cdot d\vec r = 0$

<u>Why?</u>

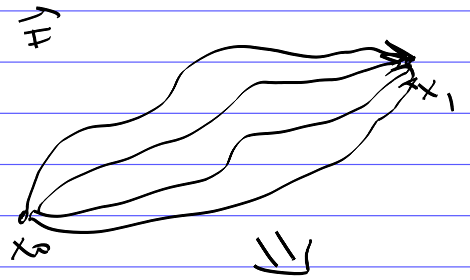


$$\int_C \vec F\cdot d\vec r = \int_{C_1} \vec F\cdot dr + \int_{C_2} \vec F\cdot dr$$

$$= f(\vec x_1) - f(\vec x_0) + f(\vec x_0) - f(\vec x_1)$$

$$= 0.$$

Consequence: path independence:



$$\Downarrow$$

$$\int_{\text{any } C} \vec F\cdot d\vec r = f(\vec x_1) - f(\vec x_0)$$

<u>Goes the other direction</u>: <u>if</u> we integrate a vector field over a path connecting a pair of points, and we find that the integral is path independent for any pair of points and any path connecting those points on that domain, then $\vec F$ is conservative.

The way to remember this:

$\vec{F}$ conservative $\Longleftrightarrow$ path independent.

Theorem 5 (from book)

If $\vec{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$

then if $\vec{F}$ is conservative, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

(requires that f has continuous 1^{st} pds)

Idea: $\vec{F}$ conservative $\Rightarrow$ exists some potential f st $\vec{F} = \nabla f$. Then:

$$P = \frac{\partial f}{\partial x} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \underset{\text{Clairaut}}{=} \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

b.e. $Q = \frac{\partial f}{\partial y}$.