NAME (*PRINT*): ______________________________

Calculus III, Final Exam

Problem	Points
True/False	
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Problem 7	
Problem 8	
Problem 9	
+ Bonus	
Total	

Rules of the exam

- You have 1h 50 min to finish this exam.
- Show your work! – any answer without an explanation will get you zero points.
- No calculators and no formula-sheets are allowed.
- When applicable, BOX the answer.
- Do not spend more than 10 min on a problem. If you get stuck, move on to the next one.
- Do not forget to write your name.

Good luck!

True / false

Circle the right answer. You don't need to justify it.

1. (True) / (False) Directional derivative $D_{\mathbf{u}}\mathbf{f} = 1$ for $\mathbf{f} = (x, 0, 0)$ and $\mathbf{u} = (1, -1, 1)$.

2. (True) / (False) Normal vector to $z = x^2 + y^2$ at $(x, y, z) = (1, 1, 2)$ is $(2, 2, -1)$.

3. (True) / (False) In spherical coordinates the equation $\phi = \pi/3$ describes a plane.

4. (True) / (False) When the vector function $\mathbf{F}$, curve C and area S satisfy the Stokes theorem, the theorem concludes that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S \nabla \times \mathbf{F} dS$

5. (True) / (False) An irrotational vector field $\mathbf{F}$ is the one for which $\nabla \times \mathbf{F} = 0$.

6. (True) / (False) Conservative vector field $\mathbf{F}$ is the one for which $\nabla \cdot \mathbf{F} = 0$.

7. (True) / (False) If $\mathbf{F}$ is a 3-dim vector field, then $div(\mathbf{F})$ is a vector field.

8. (True) / (False) If $\mathbf{F}$ is a 3-dim vector field, then $curl(\mathbf{F})$ is a vector field.

9. (True) / (False) If $f(x, y)$ has a local maximum or minimum at (a, b) and the first order partial derivatives of $f(x, y)$ exist at (a, b), then $f_x(a, b) = 0$ **OR** $f_y(a, b) = 0$.

10. (True) / (False) The field $\mathbf{F}(x, y, z) = (\sin(y), x\cos(y), -\sin(z))$ has a sink at the point $(0, 0, 0)$.

Calculus III. Midterm. Part II.

Show your work - even the correct answer with no justification will get zero points. When appropriate - box the answer. Simplify the answers as much as possible.

1. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Compute

$$\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r}$$

2. Evaluate the integral by reversing the order of integration

$$\int_0^{\pi^{1/4}} \int_{y^2}^{\pi^{1/2}} y\cos(x^2)\,dx\,dy$$

3. Let $\mathbf{F}(x, y) = (ye^x + \sin(y))\mathbf{i} + (e^x + x\cos(y))\mathbf{j}$. Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path and compute the integral, where C is the path from $(0, 1)$ to $(5, 0)$.

4. A particle on (x, y)-plane starts at the point $(-1, -1)$, moves along a horizontal straight line to the point $(1, -1)$ and then up to the point $(1, 0)$. From this point it moves along the semicircle $y = \sqrt{1 - x^2}$ to the point $(-1, 0)$ and from there to the starting point (along a vertical straight line). Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = (5x, \frac{x^3}{3} + xy^2 + y)$.

5. Find the volume of a solid E bounded by $x^2 + y^2 + z^2 = 1$ with a removed conical section $z = \sqrt{x^2 + y^2}$.

6. Let S be a surface defined by $\mathbf{r}(u,v) = (u, u+v, u-v)$ for $u^2+v^2 \leq 1$. Compute $\iint\limits_S y^2+z^2\, dS$.

7. Find the **absolute** min and max values of $f(x, y) = x^2 + (y - 1)^2$ in the domain $D = \{(x, y) : x^2 + y^2 \leq 4\}$.
Hint: first find max/min values inside the circle, then on the boundary.

8. Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, that is, calculate the flux of $\mathbf{F}$ across S, where $\mathbf{F}(x, y, z) = (\cos(z) + xy^2)\mathbf{i} + xe^{-z}\mathbf{j} + (\sin(y) + x^2 z)\mathbf{k}$, where S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

9. Compute the integral of the $curl(\mathbf{F})$, over the surface S, where the vector-field $\mathbf{F}$ is $\mathbf{F} = (y^2, x, z^2)$, and the surface S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented downward. *(You might want to use the Stokes' theorem. If you need double-angle formulas, they are on the last page of the exam.)*

Bonus

Let $\Phi(x, y, t) = \frac{1}{4\pi\sigma t} \exp\left(\frac{-(x^2+y^2)}{2\sigma t}\right)$ for $t > 0$, with coefficient $\sigma > 0$. Show that

$$\int\int_{R^2} \Phi(x, y, t) dA = 1,$$

for all fixed values of $t > 0$.

Useful Formulas:

$$\begin{aligned}
\sin^2(t) &= \frac{1 - cos(2t)}{2} \\
\cos^2(t) &= \frac{1 + cos(2t)}{2} \\
\cos^2(t) &= 1 - \sin^2(t) \\
\sin^2(t) &= 1 - \cos^2(t) \\
\nabla^2 f &= \nabla \cdot \nabla f
\end{aligned}$$

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