Solutions to Old Final 1

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True/False.

1. *Directional derivative* $D_{\mathbf{u}}\mathbf{f} = 1$ *for* $\mathbf{f} = \langle x, 0, 0\rangle$ *and* $\mathbf{u} = \langle 1, -1, 1\rangle$.

   Solution. **False** The directional derivative is

$$\begin{aligned} D_{\mathbf{u}}\mathbf{f}(x,y,z) &= \frac{d}{dt}\mathbf{f}\left((x,y,z) + t\,\langle 1,-1,1\rangle\right)\Big|_{t=0} \\ &= \frac{d}{dt}\langle x+t, 0, 0\rangle\Big|_{t=0} = \langle 1, 0, 0\rangle \end{aligned}$$

   In particular, the directional derivative of a vector field is a vector field. (This problem goes beyond the usual scope of MATH-UA 123 Calculus III.) □

2. *Normal vector to* $z = x^2 + y^2$ *at* $(x,y,z) = (1,1,2)$ *is* $\langle 2, 2, -1\rangle$.

   Solution. **True** On a surface given by an equation $g(x,y,z) = 0$, the vector $\nabla f(x,y,z)$ is normal to the surface at (x,y,z). We have $g = x^2 + y^2 - z$, so $\nabla g = \langle 2x, 2x, -1\rangle$ is normal at (x,y,z). If $x = 1$ and $y = 1$ then $\nabla g = \langle 2, 2, -1\rangle$. □

3. *In spherical coordinates the equation* $\varphi = \pi/3$ *describes a plane.*

   Solution. **False** In spherical coordinates the equation $\varphi = \pi/3$ describes a *cone.* □

4. *When the vector function* $\mathbf{F}$*, curve* C *and surface* S *satisfy the hypotheses of Stokes's theorem, the theorem concludes that* $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\,dS$

   Solution. **False** The right-hand side has a syntax error: a vector field cannot be integrated over a surface the way a function can. The statement would be true if the right-hand side were replaced with $\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$. □

5. *An irrotational vector field* $\mathbf{F}$ *is one for which* $\nabla\times\mathbf{F} = \mathbf{0}$.

Solution. **True** This is the definition, or at least equivalent to it. □

6. *A conservative vector field* $\mathbf{F}$ *is one for which* $\nabla \cdot \mathbf{F} = 0$.

   Solution. **False** The definition of conservative vector field is that there exists f such that $\nabla f = \mathbf{F}$. It is necessary then that $\nabla \times \mathbf{F} = \mathbf{0}$, but not that $\nabla \cdot \mathbf{F} = 0$ For instance, if $f(x, y, z) = x^2 + y^2 + z^2$, then $\mathbf{F} = \nabla f = \langle 2x, 2y, 2z \rangle$ is conservative by definition, But $\nabla \cdot \nabla \mathbf{F} = 6 \neq 0$. □

7. *If* $\mathbf{F}$ *is a three-dimensional vector field, then* $\operatorname{div} \mathbf{F}$ *is a vector field.*

   Solution. **False** $\operatorname{div} \mathbf{F}$ is a *scalar* field, i.e., a function. □

8. *If* $\mathbf{F}$ *is a three-dimensional vector field, then* $\operatorname{curl} \mathbf{F}$ *is a vector field.*

   Solution. **True** This is from the definition. □

9. *If* $f(x, y)$ *has a local maximum or minimum at* (a, b) *and the first-order partial derivatives of* $f(x, y)$ *exist at* (a, b)*, then* $f_x(a, b) = 0$ *OR* $f_y(a, b) = 0$.

   Solution. The statement with "OR" replace with "AND" is equivalent to Theorem 11.7.2 on page 645 of the text, so that would be true. Since "AND" is a stronger condition than "OR", the given statement is **True** as well. □

10. *The field* $\mathbf{F}(x, y, z) = \langle \sin(y), x\cos(y), -\sin(z) \rangle$ *has a sink at the point* $(0, 0, 0)$.

    Solution. **True** A sink of a vector field $\mathbf{F}$ is a place where $\nabla \cdot \mathbf{F} < 0$. We have

    $$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\sin(y) + \frac{\partial}{\partial y}x\cos(y) - \frac{\partial}{\partial z}\sin(z) = 0 - x\sin(y) - \cos(z)$$

    which at the point $(0, 0, 0)$ is -1. □

Problem 1. *Suppose* S *and* C *satisfy the hypotheses of Stokes's Theorem and* f, g *have continuous second-order partial derivatives. Compute*

$$\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r}$$

Solution. If f and g have continuous second-order partial derivatives, then Clairaut's theorem applies and nothing strange will happen with mixed partial derivatives. Now

$$\begin{aligned}\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} &= \int_C \nabla(fg)d\mathbf{r}\\ &= \iint_S \nabla\times(\nabla(fg))\cdot d\mathbf{S}\\ &= 0\end{aligned}$$

□

Problem 2. *Evaluate the integral by reversing the order of integration:*

$$\int_0^{\pi^{1/4}}\int_{y^2}^{\pi^{1/2}} y\cos\left(x^2\right)\,dx\,dy$$

Solution. The region integrated over can be described as

$$\begin{aligned}D &= \left\{ (x,y)\ \middle|\ 0\le y\le \pi^{1/4},\ y^2\le x\le \pi^{1/2} \right\}\\ &= \left\{ (x,y)\mid 0\le x\le\sqrt{\pi},\ 0\le y\le\sqrt{x} \right\}\end{aligned}$$

(draw it to see this) So

$$\begin{aligned}\int_0^{\pi^{1/4}}\int_{y^2}^{\pi^{1/2}} y\cos\left(x^2\right)\,dx\,dy &= \int_0^{\sqrt{x}}\int_0^{\sqrt{x}} y\cos\left(x^2\right)\,dy\,dx\\ &= \int_0^{\sqrt{x}} \frac{y^2}{2}\cos\left(x^2\right)\Big|_{y=0}^{y=\sqrt{x}}\,dx\\ &= \frac{1}{2}\int_0^{\sqrt{x}} x\cos\left(x^2\right)\,dx\end{aligned}$$

Substitute $u = x^2$ and $du = 2x\,dx$. Then

$$= \frac{1}{4}\int_0^{\pi}\cos(u)\,du = 0$$

□

Problem 3. *Let* $\mathbf{F}(x,y) = (ye^x + \sin(y))\mathbf{i} + (e^x + x\cos(y))\mathbf{j}$. *Show that* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *is independent of path and compute the integral, where* C *is the path from* $(0,1)$ *to* $(5,0)$.

Solution. Let $P = ye^x + \sin(y)$ and $Q = e^x + x\cos(y)$. Then

$$\frac{\partial P}{\partial y} = e^x + \cos(y) = \frac{\partial Q}{\partial x}$$

So $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ has $\int_C \mathbf{F} \cdot d\mathbf{r}$ independent of path; hence $\mathbf{F}$ is conservative. In fact $\mathbf{F} = \nabla f$, where $f(x,y) = ye^x + x\sin(y)$. So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(5,0) - f(0,1) = 0 - 1 = -1$$

□

Problem 4. *A particle on the (x,y)-plane starts at the point $(-1,-1)$, moves along a horizontal straight line to the point $(1,-1)$ and then up to the point $(1,0)$. From this point it moves along the semicircle $y = \sqrt{1-x^2}$ to the point $(-1,0)$ and from there to the starting point along a vertical straight line. Use Green's Theorem to find the work done on this particle by the force field* $\mathbf{F} = \left\langle 5x, \frac{x^3}{3} + xy^2 + y \right\rangle$.

Solution. Let D be the region enclosed by C. Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left[5x\,dx + \left(\frac{x^3}{3} + xy^2 + y \right) dy \right] \overset{\text{Green}}{=} \iint_D \left(x^2 + y^2\right) dA$$

Now $D = D_1 \cup D_2$, where

$$D_1 = \{ (x,y) \mid x^2 + y^2 \le 1,\ y \ge 0 \}$$
$$D_2 = \{ (x,y) \mid -1 \le x \le 1 \ -1 \le y \le 0 \}$$

Then using iterated integrals

$$\begin{aligned}
\iint_{D_2} \left(x^2 + y^2\right) dA &= \int_{-1}^{1} \int_{-1}^{0} \left(x^2 + y^2\right) dy\,dx \\
&= \int_{-1}^{1} x^2 y + \frac{y^3}{3} \bigg|_{y=-1}^{y=0} dx \\
&= \int_{-1}^{1} \left(x^2 + \frac{1}{3} \right) dx \\
&= 2 \int_{0}^{1} \left(x^2 + \frac{1}{3} \right) dx \\
&= 2 \left[\frac{x^3}{3} + \frac{x}{3} \right]_0^1 \\
&= \frac{4}{3}
\end{aligned}$$

For the integral over D_1 we use polar coordinates:

$$\begin{aligned}
\iint_{D_2} \left(x^2 + y^2\right) dA &= \int_0^{\pi} \int_0^1 r^2\, r\, dr\, d\theta \\
&= \int_0^{\pi} d\theta \cdot \int_0^1 r^3\, dr \\
&= \pi \cdot \frac{r^4}{4} \bigg|_0^1 = \frac{\pi}{4}
\end{aligned}$$

So

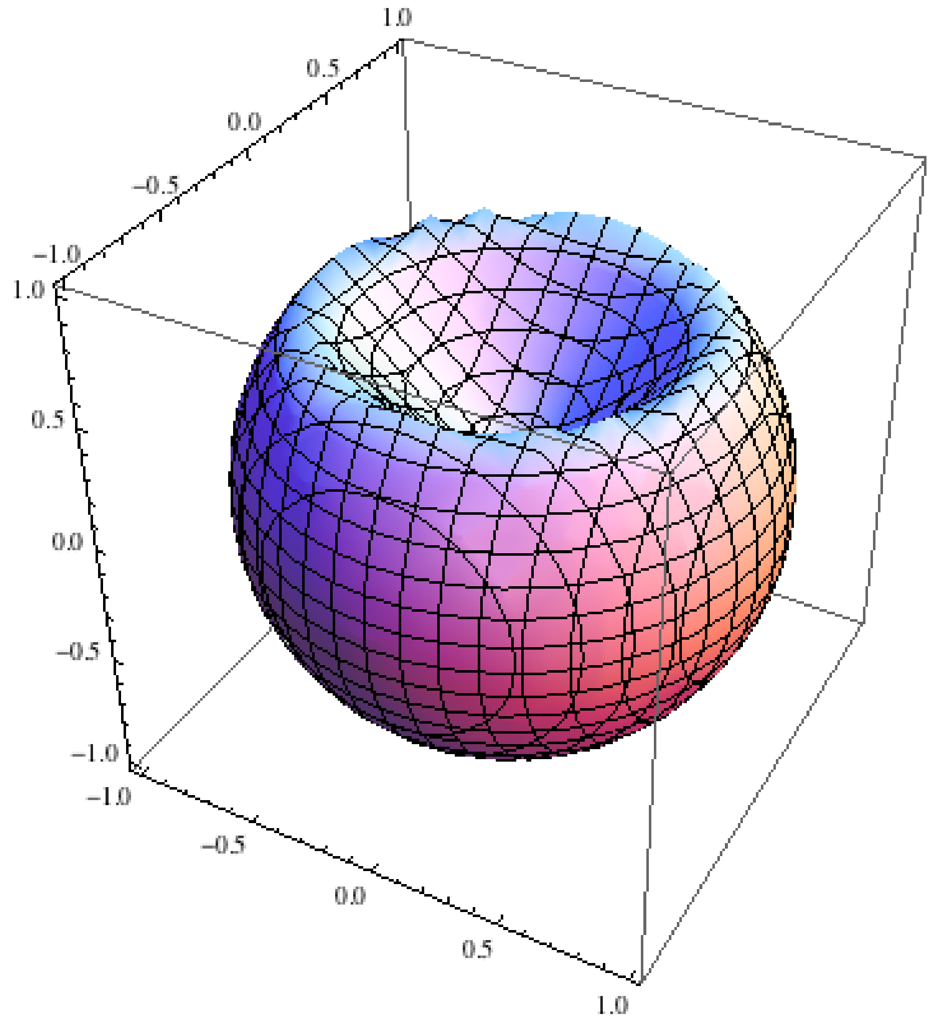
$$W = \iint_{D_1} \left(x^2 + y^2\right)\, dA + \iint_{D_2} \left(x^2 + y^2\right)\, dA = \frac{4}{3} + \frac{\pi}{4}$$

□

Problem 5. *Find the volume of the solid E bounded by $x^2 + y^2 + z^2 = 1$ with a removed conical section $z = \sqrt{x^2 + y^2}$.*

Solution. In spherical coordinates the solid E can be described as

$$E = \left\{ (\rho, \theta, \varphi) \;\middle|\; 0 \le \theta \le 2\pi,\ 0 \le \rho \le 1,\ \frac{\pi}{4} \le \theta \le \pi \right\}$$



So

$$\begin{aligned}
\mathrm{Vol}(E) &= \iiint_E dV = \int_{\pi/4}^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 \sin(\varphi)\ d\rho\, d\theta\, d\varphi \\
&= \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^2\, d\rho \cdot \int_{\pi/4}^{\pi} d\varphi \\
&= 2\pi \cdot \frac{1}{3}\left[-\cos(\varphi)\right]_{\pi/4}^{\pi} = \frac{2\pi}{3}\left[\cos(\varphi)\right]_{\pi}^{\pi/4} \\
&= \frac{2\pi}{3}\left[\frac{\sqrt{2}}{2} - (-1)\right] \\
&= \frac{2\pi}{3}\left[\frac{\sqrt{2}}{2} + \frac{2}{2}\right] \\
&= \frac{\pi}{3}\left(\sqrt{2}+2\right)
\end{aligned}$$

□

Problem 6. *Let S be the surface defined by $\mathbf{r}(u,v) = \langle u, u+v, u-v\rangle$ for $u^2 + v^2 \le 1$. Compute $\iint_S (y^2+z^2)\, dS$.*

Solution. To find the area element we compute

$$\begin{aligned}
\mathbf{r}_u &= \langle 1,1,1\rangle \\
\mathbf{r}_v &= \langle 0,1,-1\rangle \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle -2,1,1\rangle \\
|\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{6}
\end{aligned}$$

Let $D = \left\{\, (u,v) \;\middle|\; u^2+v^2 \le 1 \,\right\}$. Then

$$\begin{aligned}
\iint_S (y^2+z^2)\, dS &= \iint_D \left((u+v)^2 + (u-v)^2\right)\ \sqrt{6} dA_{u,v} \\
&= \sqrt{6}\iint_D \left(2u^2+2v^2\right)\ dA_{u,v} \\
&= 2\sqrt{6}\int_0^{2\pi}\int_0^1 r^2\, r\, dr\, d\theta \\
&= 2\sqrt{6}\int_0^{2\pi} d\theta \cdot \int_0^1 r^3\, dr \\
&= 2\sqrt{6}\cdot 2\pi \cdot \frac{1}{4} = \sqrt{6}\pi
\end{aligned}$$

□

Problem 7. *Find the* ***absolute*** *min and max values of $f(x,y) = x^2 + (y-1)^2$ in the domain $D = \left\{\, (x,y) \;\middle|\; x^2+y^2 \le 4 \,\right\}$.*

Solution. The critical points within D are the solutions to $\nabla f = \mathbf{0}$. Now

$$\frac{\partial f}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2(y-1)$$

So $\nabla f = \mathbf{0}$ implies $x = 0$ and $y = 1$. The corresponding critical value is $f(0,1) = 0$.

The critical points on the boundary of D are the solutions to $\nabla f = \lambda \nabla g$, where $g(x,y) = x^2 + y^2$. In components:

$$2x = \lambda 2x \qquad \text{and} \qquad 2(y-1) = \lambda 2y$$

The first equation implies $x = 0$ or $\lambda = 1$. If $x = 0$, then $y = \pm 2$ and there are values of λ which make the second equation consistent with the first. So we have found two critical points $(0,2)$ and $(0,-2)$, with corresponding critical values $f(0,2) = 1$ and $f(0,-2) = 9$.

If $\lambda = 1$, then the second equation reduces to $0 = -2$, a contradiction. So there are no more critical points.

The largest value of f on this domain is therefore $f(0,-2) = 9$, and the smallest value $f(0,1) = 0$. □

Problem 8. *Use the Divergence Theorem to calculate the surface integral* $\iint_S \mathbf{F} \cdot d\mathbf{S}$*; that is, calculate the flux of* $\mathbf{F}$ *across* S*, where*

$$\mathbf{F}(x,y,z) = \left(\cos(z) + xy^2\right)\mathbf{i} + xe^{-z}\mathbf{j} + \left(\sin(y) + x^2 z\right)\mathbf{k}$$

and S *is the surface of the solid bounded by the paraboloid* $z = x^2 + y^2$ *and the plane* $z = 4$*.*

Solution. Let E be the solid described by $\left\{ (x,y,z) \mid 0 \le z \le x^2 + y^2,\ 0 \le z \le 4 \right\}$ The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(\cos(z) + xy^2\right) + \frac{\partial}{\partial y} xe^{-z} + \frac{\partial}{\partial z}\left(\sin(y) + x^2 z\right) = y^2 + x^2$$

So by the divergence theorem

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_E \left(x^2 + y^2\right)\, dV \\
&= \int_0^{2\pi} \int_0^2 \int_0^4 r^2\, r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^2 r^3\, dr \cdot \int_0^4 dz \\
&= 2\pi \cdot 4 \cdot 4 = 32\pi.
\end{aligned}$$

□

Problem 9. *Compute the integral of* curl $\mathbf{F}$ *over the surface* S*, where the vector field* $\mathbf{F}$ *is* $\langle y^2, x, z^2 \rangle$*, and the surface* S *is the part of the paraboloid* $z = x^2 + y^2$ *that lies below the plane* $z = 1$*, oriented downward.*

Solution. By Stokes's Theorem,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial S} \left(y^2\,dx + x\,dy + z^2\,dz\right)$$

The curve ∂S is the unit circle in the plane $z = 1$, but the induced orientation is the negative (clockwise from above) orientation. We parametrize this curve with $x = \cos(t)$, $y = -\sin(t)$, $z = 1$. Then $dx = -\sin(t)\ dt$, $dy = -\cos(t)\ dt$, and $dz = 0$. So

$$\begin{aligned}
\int_{\partial S} \left(y^2\,dx + x\,dy + z^2\,dz\right) &= \int_0^{2\pi} \left[(-\sin(t))^2 \sin(t) + \cos(t)\left(-\cos(t)\right) + 1^2(0)\right]\ dt \\
&= -\cancelto{0}{\int_0^{2\pi} \sin(t)^3\ dt} - \int_0^{2\pi} \cos(t)^2\ dt
\end{aligned}$$

The first integral is zero by the periodicity of $\sin(t)^3$. The second integral can be computed with the double-angle trigonometric identities:

$$\begin{aligned}
-\int_0^{2\pi} \cos(t)^2\ dt &= -\frac{1}{2}\int_0^{2\pi} \left(1 + \cos(2t)\right)\ dt \\
&= -\frac{1}{2} \cdot 2\pi - \frac{1}{2}\cancelto{0}{\int_0^{2\pi} \cos(2t)\ dt}
\end{aligned}$$

again by symmetry. Hence

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = -\pi$$

□

Bonus. *Let* $\Phi(x, y, t) = \frac{1}{4\pi\sigma t}\exp\left(-\frac{x^2+y^2}{2\sigma t}\right)$ *for* $t > 0$*,* $\sigma > 0$*. Find*

$$\iint_{\mathbb{R}^2} \Phi(x, y, t)\,dA_{x,y}$$

Solution. This is an improper integral, which we can switch to polar:

$$\begin{aligned}
\iint_{\mathbb{R}^2} \Phi(x, y, t)\,dA_{x,y} &= \frac{1}{4\pi\sigma t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2 + y^2}{2\sigma t}\right)\,dx\,dy \\
&= \frac{1}{4\pi\sigma t}\int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma t}\right)\,r\,dr\,d\theta \\
&= \frac{2\pi}{4\pi\sigma t}\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma t}\right)\,r\,dr \\
&= \frac{1}{2\sigma t}\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma t}\right)\,r\,dr
\end{aligned}$$

Let $u = \frac{r^2}{2\sigma t}$, and $du = \frac{2r\,dr}{2\sigma t}$. So $r\,dr = \sigma t\,du$. Then

$$
\begin{aligned}
&= \frac{1}{2}\int_0^\infty \exp(-u)\;du \\
&= -\frac{1}{2}\exp(-u)\Big|_0^\infty = \frac{1}{2}.
\end{aligned}
$$

□