Free Response

Be sure to show all your work neatly and indicate your final answer where appropriate.

FR1 (15 points)

(a) (7 points) Let L be the line given by $\mathbf{r}(t) = -2\,\mathbf{i} + t\,\mathbf{j} - t\,\mathbf{k}$. Find the line containing the point $(1,4,2)$ and perpendicular to L (lines must intersect to be considered perpendicular).

Solution:

$$\text{proj}_{\langle 0,1,-1\rangle}(\langle 1,4,2\rangle - \langle -2,0,0\rangle) = \langle 0,1,-1\rangle$$

Thus direction of the line is

$$\langle 3,4,2\rangle - \langle 0,1,-1\rangle$$

and the line is simply

$$\mathbf{r}(t) = \langle 1+3t, 4+3t, 2+3t\rangle$$

(b) (8 points) Use the method of Lagrange multipliers to find the point on the line $2x+3y=6$ closest to the origin.

Solution:

Use distance squared instead. $f(x,y) = x^2+y^2$ $\quad g(x,y) = 2x+3y = 6$.

(4 points) Set up Lagrange's equations $\nabla f = \lambda \nabla g$.

$$\begin{aligned} 2x &= 2\lambda \\ 2y &= 3\lambda \\ 2x+3y &= 6 \end{aligned}$$

(4 point) Solve it.

$$2y = 3x \implies 2x + \frac{9}{2}x = 6 \implies x = \frac{12}{13}, y = \frac{18}{13}.$$

FR2 (15 points) Let C be a piecewise smooth curve from $(0,0)$ to $(\pi/2,0)$. Evaluate the following line integral

$$\int_C \cos x \cos y \, dx + (1 - \sin x \sin y) \, dy$$

two ways:

(a) (8 points) by using the Fundamental Theorem of Line Integrals.

Solution:

Solve $\nabla f = \langle \cos x \cos y, 1 - \sin x \sin y \rangle$.

$$f(x,y) = y + \sin x \cos y + C$$

FTLI says

$$\int_C \nabla f \cdot d\mathbf{r} = f(\pi/2, 0) - f(0,0) = 1$$

(b) (7 points) by directly computing the integral along a convenient path. Solution:

$$\mathbf{r}(t) = t\,\mathbf{i} \qquad 0 \le t \le \frac{\pi}{2}$$

$$\int_C P\,dx + Q\,dy = \int_a^b P(x(t), y(t))x'(t)\,dt + Q(x(t), y(t))y'(t)\,dt = \int_0^{\pi/2} \cos t \cos 0 dt = \sin(\pi/2) - \sin 0 = 1$$

FR3 (15 points) Let $\mathbf{F} = -z^2\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$ and let C be the triangle with vertices at $(0,0,0)$, $(3,3,0)$, and $(0,0,6)$ (oriented in that order).

Use Stokes' Theorem to compute the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Solution:

(2 points) State Stokes' Theorem correctly.

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

where S is the piece of the plane inside the triangle.

(2 points) Compute the curl of $\mathbf{F}$.

$$\operatorname{curl} \mathbf{F} = -2 - 2z\,\mathbf{j} + y\,\mathbf{k}$$

(4 points) Parameterize the surface.

$$\mathbf{r}(u,v) = \langle u, u, v \rangle$$

$$0 \le u \le 3$$
$$0 \le v \le 6 - 2u$$

(2 points) Compute $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i} - \mathbf{j}$.

(5 points) Set up and compute iterated integral in u and v.

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_0^3 \int_0^{6-2u} (2v+2)\,dv\,du = 54$$

FR4 (15 points) Let S be the upper hemisphere described by $x^2+y^2+z^2=16$ with $z \geq 0$ (oriented upward). We will use the Divergence Theorem to compute the flux of the vector field

$$\mathbf{F}(x,y,z) = (xz - yx + 2x)\,\mathbf{i} + (2y^2 - 2xy - yz)\,\mathbf{j} + (2zx - 3zy)\,\mathbf{k}$$

through S as follows.

(a) (3 points) Let D be the part of the xy-plane under the hemisphere. Find $\mathbf{n}$, the outward-pointing normal vector to D.
Solution: $\mathbf{n} = -\mathbf{k}$
Give credit for overdoing it by parameterizing the surface and computing $\mathbf{r}_u \times \mathbf{r}_v$.
The question is not very clear about orientation (unfortunately). I would be forgiving.

(b) (3 point) Compute $\mathbf{F} \cdot \mathbf{n}$ on D.
Solution:

$$-(2zx - 3zy) = 0$$

on D.

(c) (3 points) Compute $\operatorname{div} \mathbf{F}$.
Solution:

$$\nabla \cdot \mathbf{F} = (z - y + 2) + (4y - 2x - z) + (2x - 3y) = 2$$

(d) (6 points) Use the Divergence Theorem to complete the computation. Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} + \iiint_D \mathbf{F} \cdot d\mathbf{S} = \iiint_{B^+} \operatorname{div} \mathbf{F} \, dV$$

where B^+ is the upper part of the solid sphere of radius 4.
The second surface integral is 0, so the flux through S is

$$\iiint_{B^+} 2\,dV = 2\frac{1}{2}\frac{4}{3}\pi 4^3 = \frac{256}{3}\pi$$