New York University
MATH.UA 123 Calculus 3

Problem Set 1

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

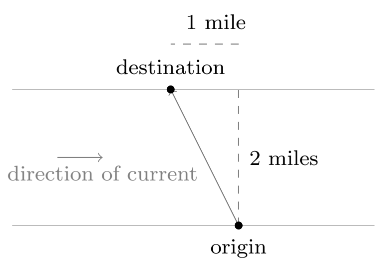
Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (3 points) Describe in words the set of points that satisfy the following two equations:

$$x^2 + y^2 + z^2 = 4y,$$

$$x = z.$$

2. (4 points) A swimmer plans to swim across a river that is 2 mile wide, aiming to land at a point 1 mile upstream from her starting point. She can swim at a constant speed of 1.5 miles per hour. The current in the river flows at 0.5 mile per hour downstream.
   (a) In order to swim in a straight line to reach her destination, in what direction should she steer?
   (b) How long will the trip take?



3. (3 points) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Show that:
   (a) the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$
   (b) the vector $\mathbf{u} - \text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}$ lies in a common plane with $\mathbf{v}$ and $\mathbf{w}$

4. (4 points) Use the scalar triple product to verify that the three vectors below are coplanar (all three lie on the same plane):

$$\begin{aligned} \mathbf{u} &= \cancel{\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}} \quad \text{CORRETION: } \mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \\ \mathbf{v} &= 2\mathbf{i} + 9\mathbf{j} - \mathbf{k} \\ \mathbf{w} &= 4\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}. \end{aligned}$$

5. (4 points) Find parametric equations for the following lines:
   (a) the line that goes through the points $(0, -3, 1)$ and $(5, 2, 2)$

(b) ~~the~~ a line that goes through the point $(3, 2, 1)$ and is perpendicular to the line $\mathbf{r}_0 + t\mathbf{v}$ where $\mathbf{r}_0 = \langle 1, 0, -2\rangle$ and $\mathbf{v} = \langle -1, 1, 4\rangle$.

6. (4 points) Find equations for the following planes

   (a) the plane that passes through the point $(1, -1, 1)$ and contains the line with symmetric equations

$$x = 2y = 3z.$$

   (b) the plane that contains all points that are equidistant from the points $(3, 2, -1)$ and $(-7, 4, -3)$.

7. (4 points) (a) Find the distance between the two parallel planes:

$$2x + y - 3z = 4$$

   and

$$4x + 2y - 6z = -2.$$

   (b) Suppose that $\mathbf{m} \cdot \mathbf{r} = a$ and $\mathbf{n} \cdot \mathbf{r} = b$ describe two parallel planes. Derive a formula for the distance between them. Your answer will be in terms of $\mathbf{m}, \mathbf{n}, a, b$.

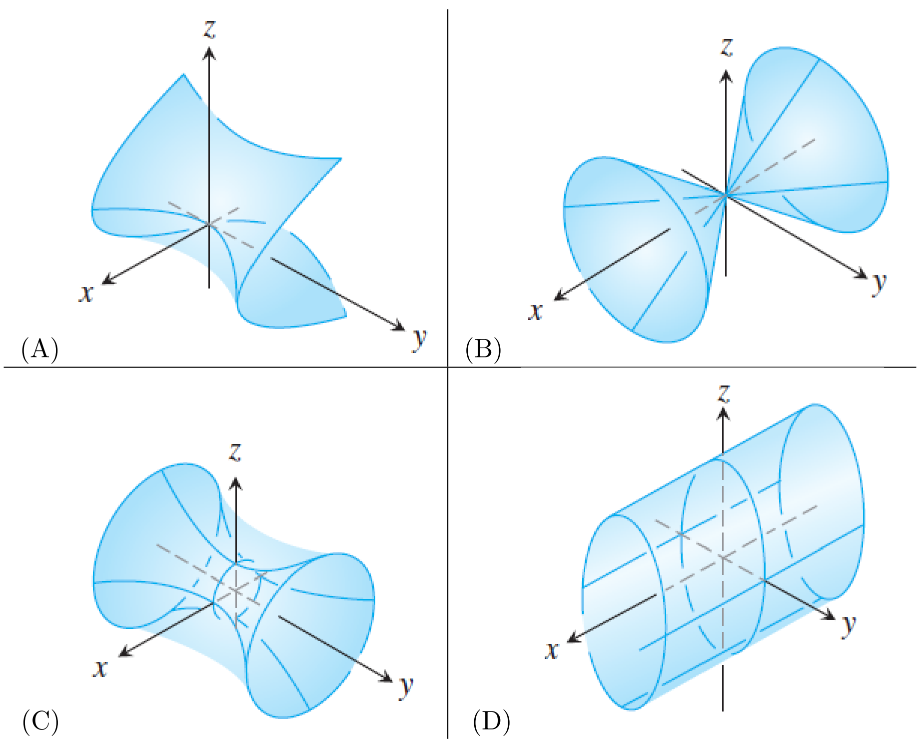
8. (4 points) (1) Match the equation with the surface it defines and (2) identify each surface by type (ellipsoid, paraboloid, etc.).

   (a) $9y^2 + z^2 = 16$

   (b) $x = y^2 - z^2$

   (c) $x^2 = y^2 + z^2$

   (d) $z^2 + x^2 - y^2 = 1$



9. (3 points) Find an equation for the surface consisting of all points P for which the distance from P to the y-axis is half the distance from P to the xz-plane. Identify the surface.

10. (3 points) Show that the curve with parametric equations

$$x = \sin(t), \ y = \cos(t), \ z = \sin^2(t)$$

is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.

11. (4 points) (a) Find the point on the curve

$$\mathbf{r}(t) = \langle t^3 + 3t, t^2 + 1, \ln(1 + 2t)\rangle, \ 0 \le t \le \pi,$$

where the tangent line is ~~parallel~~ orthogonal to the plane

$$15x + 4y + 0.4z = 10.$$

(b) Find the equation of the line tangent to the curve $\mathbf{r}(t)$ at the point you found in part (a).

12. (3 points) At what point on the curve

$$x = t^3, \ y = 3t, \ z = t^4$$

is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

13. (3 points) Show that if the position vector $\mathbf{r}(t)$ is always perpendicular to the velocity vector $\mathbf{r}'(t)$, then the curve lies entirely on a sphere centered at the origin.

14. (4 points) Show that

$$\frac{d}{dt}\left[\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))\right] = \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t))$$