New York University
MATH.UA 123 Calculus 3

Problem Set 1

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (3 points) Describe in words the set of points that satisfy the following two equations:

$$x^2 + y^2 + z^2 = 4y,$$

$$x = z.$$

Solution: The first equation can be rewritten (by completing the square) as:

$$x^2 + (y-2)^2 + z^2 = 4,$$

which is the equation that describes the sphere of radius 2, centered at the point $(0, 2, 0)$.

The second equation describes a plane that passes through the y-axis, and makes a 45-degree angle with the z- axis and with the x-axis.

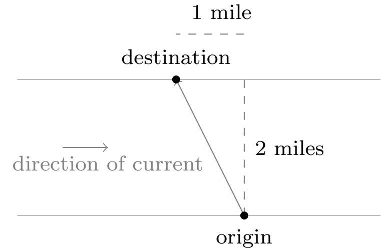
So, the set of points that satisfy both equations is the intersection of the sphere and the plane that are described above.

(Since the point $(0, 2, 0)$, the center of the sphere, lies on the plane $x = z$, we could further note that this intersection is in fact a circle of radius 2 centered at $(0, 2, 0)$, lying on the plane $x = z$.)

2. (4 points) A swimmer plans to swim across a river that is 2 mile wide, aiming to land at a point 1 mile upstream from her starting point. She can swim at a constant speed of 1.5 miles per hour. The current in the river flows at 0.5 mile per hour downstream.

   (a) In order to swim in a straight line to reach her destination, in what direction should she steer?

   (b) How long will the trip take?



Solution:

(a) Let the swimmer's point of origin be denoted by $O = (0,0)$ and her destination by $D = (-1,2)$. The swimmer's desired trajectory points in a direction described by the vector $\overrightarrow{OD} = \langle -1, 2\rangle$.

The velocity (magnitude and direction) of the current can be described by the vector $\mathbf{c} = \langle 0.5, 0\rangle$. Let $\mathbf{s} = \langle x, y\rangle$ denote the velocity that the swimmer should steer. This means that the magnitude of $\mathbf{s}$ must be 1.5:

$$\sqrt{x^2 + y^2} = 1.5.$$

Therefore, expressing x in terms of y: $\mathbf{s} = \langle -\sqrt{2.25 - y^2}, y\rangle$ (since we know the swimmer wants to steer to the left).

We want to solve for y above (to determine the direction the swimmer should steer), such that: The relationship between the desired trajectory $\overrightarrow{OD}$, the current direction $\mathbf{c}$, and the swimmer's steering direction $\mathbf{s}$ is

$$t\,(\mathbf{c} + \mathbf{s}) = \overrightarrow{OD}$$

or:

$$\mathbf{c} + \mathbf{s} = \frac{1}{t}\overrightarrow{OD},$$

$$\langle 0.5, 0\rangle + \langle -\sqrt{2.25 - y^2}, y\rangle = \frac{1}{t}\langle -1, 2\rangle,$$

where t denote the time it takes for the swimmer to reach her destination.

Writing the x- and y-components separately:

$$0.5 - \sqrt{2.25 - y^2} = -1/t \qquad (1)$$

$$y = 2/t. \qquad (2)$$

So, substituting $y = 2/t$ from equation 2, equation 1 above can be written as:

$$0.5 - \sqrt{2.25 - y^2} = -y/2$$

$$\Leftrightarrow \quad 2.25 - y^2 = (-y/2 - 0.5)^2$$

$$\Leftrightarrow \quad 0 = 1.25y^2 + 0.5y - 2.$$

Using the quadratic formula, and knowing that y must be positive, we obtain:

$$y = \frac{1}{5}(\sqrt{41} - 1) \approx 1.08.$$

So, $x = -\sqrt{2.25 - \frac{1}{25}(42 - 2\sqrt{41})} \approx -1.04$. Therefore, the swimmer should steer in the direction $\mathbf{s} \approx \langle -1.04, 1.08\rangle$.

(b) From 2, we know that $t = \frac{2}{y}$. So, $t = \frac{2}{\frac{1}{5}(\sqrt{41}-1)} \approx 1.85$ hours.

3. (3 points) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Show that:

(a) the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$

(b) the vector $\mathbf{u} - \text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}$ lies in a common plane with $\mathbf{v}$ and $\mathbf{w}$

Solution:

(a) Note that

$$\begin{aligned}\mathbf{v}\cdot(\mathbf{u}-\text{proj}_{\mathbf{v}}\mathbf{u}) &= \mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\left(\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{v}||^2}\mathbf{v}\right)\\ &= \mathbf{v}\cdot\mathbf{u}-\frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{v}||^2}(\mathbf{v}\cdot\mathbf{v})\\ &= \mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{u}=0,\end{aligned}$$

indicating that $\mathbf{v}$ and $(\mathbf{u}-\text{proj}_{\mathbf{v}}\mathbf{u})$ are orthogonal to one another.

(b) Similarly, note that

$$\begin{aligned}(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{u}-\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}) &= (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}-(\mathbf{v}\times\mathbf{w})\cdot\left(\frac{(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}}{||\mathbf{v}\times\mathbf{w}||^2}(\mathbf{v}\times\mathbf{w})\right)\\ &= (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}-\frac{(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}}{||(\mathbf{v}\times\mathbf{w})||^2}((\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w}))\\ &= (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}-(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}=0,\end{aligned}$$

indicating that $\mathbf{u}-\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}$ lies on a plane that is orgthogonal to $\mathbf{v}\times\mathbf{w}$. However, we know that both $\mathbf{v}$ and $\mathbf{w}$ also lie on a plane that is orthogonal to $\mathbf{v}$ and to $\mathbf{w}$. Therefore, $\mathbf{u}-\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ all lie in the same plane.

4. (4 points) Use the scalar triple product to verify that the three vectors below are coplanar (all three lie on the same plane):

$$\begin{aligned}\mathbf{u} &= \sout{\mathbf{i}-2\mathbf{j}+4\mathbf{k}} \quad \text{CORRETION: } \mathbf{u}=\mathbf{2}\mathbf{i}-2\mathbf{j}+4\mathbf{k}\\ \mathbf{v} &= 2\mathbf{i}+9\mathbf{j}-\mathbf{k}\\ \mathbf{w} &= 4\mathbf{i}+7\mathbf{j}+3\mathbf{k}.\end{aligned}$$

Solution: We can compute that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=0$. This indicates that $\mathbf{u}$ is orthogonal to the vector $\mathbf{v}\times\mathbf{w}$, which itself is orthogonal to the plane that contains $\mathbf{v}$ and $\mathbf{w}$. Thus, the three vectors lie on the same plane.

5. (4 points) Find parametric equations for the following lines:

(a) the line that goes through the points $(0,-3,1)$ and $(5,2,2)$

(b) ~~the~~ a line that goes through the point $(3,2,1)$ and is perpendicular to the line $\mathbf{r}_0+t\mathbf{v}$ where $\mathbf{r}_0=\langle 1,0,-2\rangle$ and $\mathbf{v}=\langle -1,1,4\rangle$.

Solution:

(a) First, we find the vector that goes from the first to the second point:

$$\mathbf{v}=\langle 5-0, 2-(-3), 2-1\rangle=\langle 5,5,1\rangle.$$

Hence, the line that goes in this direction, and through the point $(0,-3,1)$ (and the point $(5,2,2)$) is

$$\mathbf{r}=\langle 0,-3,1\rangle+t\langle 5,5,1\rangle.$$

Equivalently, a set of parametric equations for the line:

$$x = 5t,\ y = -3 + 5t,\ z = 1 + t.$$

(b) (There are lots of lines that goes through this point, whose direction is perpendicular to the direction of the line $\mathbf{r}_0 + t\mathbf{v}$. The following is one of them.)

Call the point $(3, 2, 1)$ Q.

Note that the line $\mathbf{r}_0 + t\mathbf{v}$ goes through the point $(1, 0, -2)$. Call this point P.

The projection of the vector $\overrightarrow{PQ} = \langle 2, 2, -3\rangle$ onto $\mathbf{v} = \langle -1, 1, 4\rangle$ is

$$\mathrm{proj}_{\mathbf{v}}\overrightarrow{PQ} = \frac{-2 + 2 + 12}{1 + 1 + 16}\langle -1, 1, 4\rangle = \frac{2}{3}\langle -1, 1, 4\rangle.$$

Recall (from problem 1(a)) that the vector

$$\overrightarrow{PQ} - \mathrm{proj}_{\mathbf{v}}\overrightarrow{PQ} = \frac{2}{3}\langle 8, 4, 1\rangle$$

is orthogonal to $\mathbf{v}$, so we can use this as the direction vector of the desired line that goes through the point $(3, 2, 1)$:

$$\mathbf{r} = \langle 3, 2, 1\rangle + t\langle 8/3, 4/3, 1/3\rangle.$$

Equivalently, the parametric equations are:

$$x = 3 + \frac{8}{3}t,\ y = 2 + \frac{4}{3}t,\ z = 1 + \frac{1}{3}t.$$

6. (4 points) Find equations for the following planes

(a) the plane that passes through the point $(1, -1, 1)$ and contains the line with symmetric equations

$$x = 2y = 3z.$$

(b) the plane that contains all points that are equidistant from the points $(3, 2, -1)$ and $(-7, 4, -3)$.

Solution:

(a) The line with the symmetric equations $x = 2y = 3z$ can be rewritten as:

$$x = t,\ y = \frac{1}{2}t,\ z = \frac{1}{3}t,$$

or

$$\mathbf{r} = t\langle 1, 1/2, 1/3\rangle.$$

So, this line points in the direction $\mathbf{v} = \langle 1, 1/2, 1/3\rangle$ and goes through the origin.

The vector from the origin to the point $(1, -1, 1)$ is $\mathbf{u} = \langle 1, -1, 1\rangle$. We can find a vector that is orthogonal to the desired plane by taking the cross product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = \langle 5/6, -2/3, -3/2\rangle.$$

Thus, the equation of the plane:

$$\langle 5/6, -2/3, -3/2\rangle \cdot \langle x, y, z\rangle = 0,$$

or

$$\frac{5}{6}x - \frac{2}{3}y - \frac{3}{2}z = 0.$$

(b) One point that is equidistant from $A = (3, 2, 1)$ and $B = (-7, 4, -3)$ is the midpoint of the line segment $\overline{AB}$, namely the point $(-2, 3, -2)$. Furthermore, the plane containing all such points must be perpendicular to the direction vector $\overrightarrow{AB} = \langle -10, 2, -4\rangle$. So, the equation of the plane:

$$\langle -10, 2, -4\rangle \cdot (\langle x, y, z\rangle - \langle -2, 3, -2\rangle) = 0,$$

or

$$-10x + 2y - 4z = 34.$$

7. (4 points) (a) Find the distance between the two parallel planes:

$$2x + y - 3z = 4$$

and

$$4x + 2y - 6z = -2.$$

(b) Suppose that $\mathbf{m} \cdot \mathbf{r} = a$ and $\mathbf{n} \cdot \mathbf{r} = b$ describe two parallel planes. Derive a formula for the distance between them. Your answer will be in terms of $\mathbf{m}, \mathbf{n}, a, b$.

Solution:

(a) We will find a point in the first plane and a point on the second plane. For example, $A = (2, 0, 0)$ lies on the first plane and $B = (0, -1, 0)$ likes on the second. Then, we project the vector $\overrightarrow{AB} = \langle -2, -1, 0\rangle$ onto the normal vector of the first plane (which is a scalar multiple of the normal vector of the second plane), namely $\mathbf{n} = \langle 2, 1, -3\rangle$. The length of this projection is the distance between the two planes:

$$\begin{aligned} \text{comp}_{\mathbf{n}}\overrightarrow{AB} &= |\text{proj}_{\mathbf{n}}\overrightarrow{AB}| \\ &= \frac{|\mathbf{n} \cdot \overrightarrow{AB}|}{|\mathbf{n}|} \\ &= \frac{|-4-1+0|}{\sqrt{4+1+9}} = \frac{5}{\sqrt{14}}. \end{aligned}$$

(b) First note that if $\mathbf{m}$ and $\mathbf{n}$ are the normal vectors of two parallel planes, then they must be scalar multiples of one another: $\mathbf{m} = \alpha\mathbf{n}$ for some scalar α.

As in part (a), we start by considering any point $A = (x_1, y_1, z_1)$ from the first plane and any point $B = (x_2, y_2, z_2)$ from the second plane. (This means that $x_1, y_1, z_1, x_2, y_2, z_2$ must satisfy:

$$\mathbf{n} \cdot \langle x_1, y_1, z_1\rangle = a, \text{ and } \mathbf{m} \cdot \langle x_2, y_2, z_2\rangle = b.)$$

As in part (a), the distance between the two planes are:

$$\begin{aligned}
\text{comp}_{\mathbf{n}}\overrightarrow{AB} &= |\text{proj}_{\mathbf{n}}\overrightarrow{AB}| \\
&= \frac{|\mathbf{n}\cdot\overrightarrow{AB}|}{|\mathbf{n}|} \\
&= \frac{|\mathbf{n}\cdot\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle|}{|\mathbf{n}|} \\
&= \frac{|\mathbf{n}\cdot\langle x_2, y_2, z_2\rangle - \mathbf{n}\cdot\langle x_1, y_2, z_1\rangle|}{|\mathbf{n}|} \\
&= \frac{|\alpha\mathbf{m}\cdot\langle x_2, y_2, z_2\rangle - a|}{|\mathbf{n}|} \\
&= \frac{|\alpha b - a|}{|\mathbf{n}|}
\end{aligned}$$

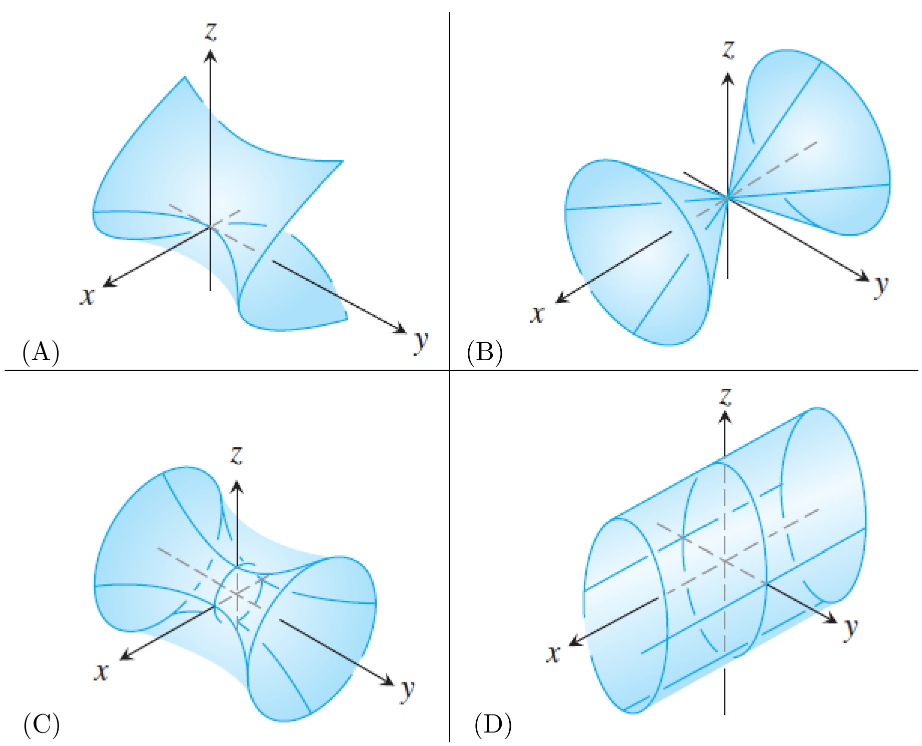
8. (4 points) (1) Match the equation with the surface it defines and (2) identify each surface by type (ellipsoid, paraboloid, etc.).

   (a) $9y^2 + z^2 = 16$

   (b) $x = y^2 - z^2$

   (c) $x^2 = y^2 + z^2$

   (d) $z^2 + x^2 - y^2 = 1$



Solution:

(a) $9y^2 + z^2 = 16$ is a cylinder, whose cross-section parallel to the yz-plane is an ellipse. Thus, this equation corresponds to surface (D), an elliptic cylinder.

(b) $x = y^2 - z^2$. The trace for $x = k$ is a hyperbola, the trace for $y = k$ is a parabola, and the trace for $z = k$ is also a parabola. In particular, for each fixed y, as z goes to $+\infty$ or $-\infty$, x goes to $-\infty$. For each fixed z, as y goes to $+\infty$ or $-\infty$, x goes to $+\infty$. Thus, this equation corresponds to surface (A), a hyperbolic paraboloid.

(c) $x^2 = y^2 + z^2$. The trace for $x = k$ is a circle, the trace for $y = k$ is a hyperbola, and the trace for $z = k$ is also a hyperbola. Therefore, the resulting surface is the cone (B).

(d) $z^2 + x^2 - y^2 = 1$. The trace for $x = k$ is a hyperbola, the trace for $y = k$ is a circle, and the trace for $z = k$ is a hyperbola. Therefore, the corresponding surface is (C), a hyperboloid of one sheet.

9. (3 points) Find an equation for the surface consisting of all points P for which the distance from P to the y-axis is half the distance from P to the xz-plane. Identify the surface.

Solution: The distance of a point (x, y, z) to the xy-plane is $|y|$ and its distance to the y axis is $\sqrt{x^2 + z^2}$. Therefore, a point (x, y, z) is on the given surface if and only if

$$\frac{1}{2}|y| = \sqrt{x^2 + z^2}$$

or

$$\frac{y^2}{4} = x^2 + z^2.$$

This equation corresponds to a cone, whose traces in $y = k$ (parallel to the xz-plane) are circles of radius $|k|/2$.

10. (3 points) Show that the curve with parametric equations

$$x = \sin(t), \ y = \cos(t), \ z = \sin^2(t)$$

is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.

Solution: We can check that $x = \sin(t), \ y = \cos(t), \ z = \sin^2(t)$ satisfies both equations:

$$z = x^2 \ \text{ and } x^2 + y^2 = 1.$$

This shows that the curve lies on both surfaces.

We also need to show that the intersection of the two surfaces do not contain any points that do not lie on the given curve. Since $z = x^2$, we know that points on the intersection of these surfaces cannot have a negative z component. Since $x^2 + y^2 = 1$, we also know that the x and y components must lie between -1 and 1. So, we can let $x = \sin(t)$. This means that z must equal $\sin^2(t)$ and $y = \cos(t)$.

11. (4 points) (a) Find the point on the curve

$$\mathbf{r}(t) = \langle t^3 + 3t, t^2 + 1, \ln(1 + 2t)\rangle, \ 0 \le t \le \pi,$$

where the tangent line is ~~parallel~~ orthogonal to the plane

$$15x + 4y + 0.4z = 10.$$

(b) Find the equation of the line tangent to the curve $\mathbf{r}(t)$ at the point you found in part (a).

Solution:

(a) The derivative vector is

$$\mathbf{r}'(t) = \left\langle 3t^2 + 3, 2t, \frac{2}{1+2t} \right\rangle,$$

and this direction is tangent to the curve $\mathbf{r}(t)$.

The given plane has as a normal vector, the vector $\langle 15, 4, 0.4\rangle$. Note that $\mathbf{r}'(2) = \langle 15, 4, 0.4\rangle$, which means that the line tangent to the curve at the point $\mathbf{r}(2) = \langle 14, 5, \ln(5)\rangle$ is orthogonal to the plane.

(b) The equation of the tangent line:

$$\langle 14, 5, \ln(5)\rangle + t\langle 15, 4, 0.4\rangle.$$

12. (3 points) At what point on the curve

$$x = t^3, \ y = 3t, \ z = t^4$$

is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

Solution: Note that

$$\mathbf{r}'(t) = \langle 3t^2, 3, 4t^3\rangle.$$

The normal plane is the plane that is orthogonal to the tangent vector above. So, the normal plane is parallel to the plane $6x + 6y - 8z = 1$ if the tangent vector above is parallel to the normal vector of this plane. That is, if

$$\langle 3t^2, 3, 4t^3\rangle = \alpha \ \langle 6, 6, -8\rangle$$

for some scalar α.

Letting $\alpha = 0.5$ (in order to make the y-component of the left-hand side equal to that of the right-hand side),

$$\langle 3t^2, 3, 4t^3\rangle = \langle 3, 3, -4\rangle.$$

So, $3t^2 = 3$ and $4t^3 = -4$, which means that t must be -1.

So, the normal plane of $\mathbf{r}(t)$ is parallel to the given plane at $\mathbf{r}(1) = \langle 1, 3, 1\rangle$.

13. (3 points) Show that if the position vector $\mathbf{r}(t)$ is always perpendicular to the velocity vector $\mathbf{r}'(t)$, then the curve lies entirely on a sphere centered at the origin.

Solution: Note that

$$\begin{aligned} \frac{d}{dt}|\mathbf{r}(t)|^2 &= \frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t)) \\ &= \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) \\ &= 2\mathbf{r}'(t) \cdot \mathbf{r}(t). \end{aligned}$$

Since we were told that that $\mathbf{r}(t)$ is always perpendicular to the velocity vector $\mathbf{r}'(t)$, then the derivative of $|\mathbf{r}(t)|^2$ must be zero, which means that the magnitude of $\mathbf{r}(t)$ is constant. This is the same as saying that the curve $\mathbf{r}(t)$ must lie on some sphere centered at the origin.

14. (4 points) Show that

$$\frac{d}{dt}\ [\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))] = \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t))$$

Solution:

$$\begin{aligned}
\frac{d}{dt}\ [\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))] &= \mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t)) + \mathbf{r}(t) \cdot \frac{d}{dt}(\mathbf{r}'(t) \times \mathbf{r}''(t)) \\
&\quad \text{(product rule for the dot product of two vector functions)} \\
&= (\mathbf{r}'(t) \times \mathbf{r}'(t)) \cdot \mathbf{r}''(t) + \mathbf{r}(t) \cdot \frac{d}{dt}(\mathbf{r}'(t) \times \mathbf{r}''(t)) \\
&\quad \text{(property of triple products)} \\
&= \mathbf{0} \cdot \mathbf{r}''(t) + \mathbf{r}(t) \cdot \frac{d}{dt}(\mathbf{r}'(t) \times \mathbf{r}''(t)) \\
&\quad \text{(the cross product of parallel vectors is the zero vector)} \\
&= \mathbf{r}(t) \cdot (\mathbf{r}''(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'''(t)) \\
&\quad \text{(product rule for the cross product of two vector functions)} \\
&= \mathbf{r}(t) \cdot (\mathbf{0} + \mathbf{r}'(t) \times \mathbf{r}'''(t)) \\
&\quad \text{(the cross product of parallel vectors is the zero vector)} \\
&= \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t))
\end{aligned}$$