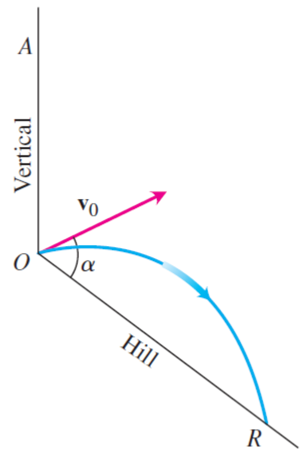
New York University
MATH.UA 123 Calculus 3

Problem Set 2

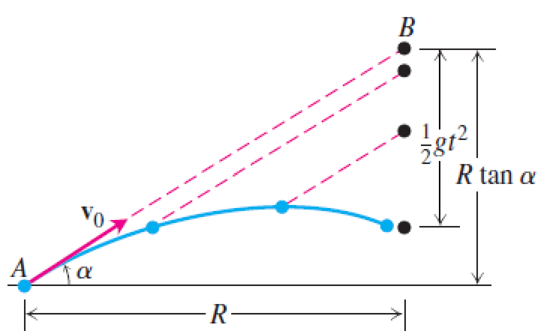
This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (4 points) A projectile is launched straight down an inclined plane as shown in he figure below. Show that the greatest downhill range (the distance from the initial position to the point where the projectile hits the ground) is achieved when the initial velocity vector bisects the angle $\angle AOR$ between the vertical line and the plane.

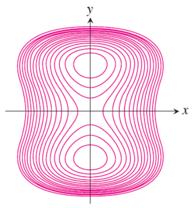


2. (4 points) The figure below shows an experiment with two marbles. Marble A was launched towards marble B with launch angle α and initial speed $v_0 = |\mathbf{v}_0| > 0$. At the same instant, marble B was released to fall from rest at $R \tan \alpha$ units directly above a spot R units horizontally downrange from A. Show that the marbles collide regardless of the initial speed v_0.

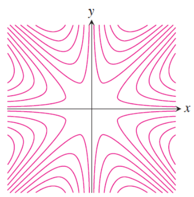


3. (3 points) Match each set of level curves with the appropriate graph of function. Briefly explain your choices.

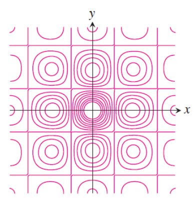
1.



2.



3.



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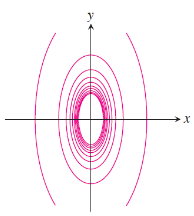

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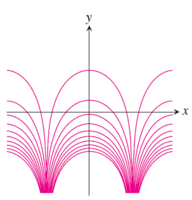
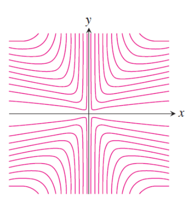
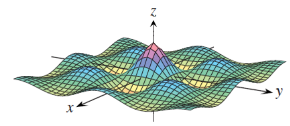


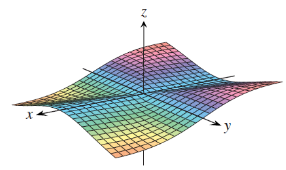
6.



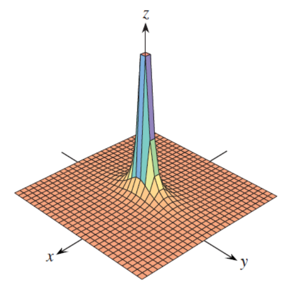
a.



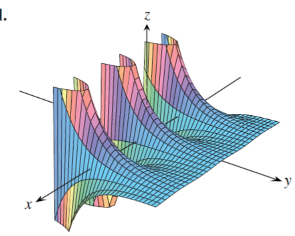
b.



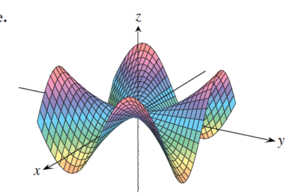
c.



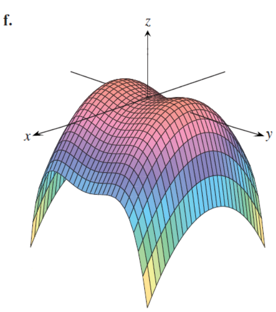
d.



e.



f.



4. (4 points) For each of the following functions: (i) find the function's domain, (ii) find the function's range, and (iii) sketch several of its level curves.

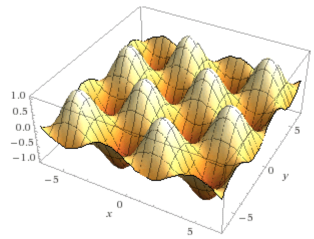
   (a) $f(x, y) = \frac{2y-x}{x+y+1}$

   (b) $f(x, y) = 3 - |x| - 4|y|$

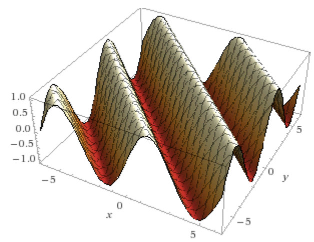
   (c) $f(x, y) = \sqrt{x^2 - y^2 - 16}$

5. (3 points) The functions $\sin(x)$ and $\cos(x)$ have wavy, periodic graphs. Manipulate these function (or devise your own) to find a function $f(x,y)$ whose graph has the following general shapes:

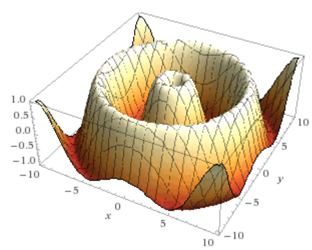
   (a) "Egg carton"



   (b) "Wavy cylinder"



   (c) "Circular wave"



6. (4 points) (a) Use the squeeze theorem to evaluate

$$\lim_{(x,y)\to(0,0)} \tan(x)\ \sin\left(\frac{1}{|x|+|y|}\right).$$

   (b) State whether function $f(x,y) = \tan(x)\ \sin\left(\frac{1}{|x|+|y|}\right)$ is continuous at $(0,0)$. Explain why or why not.

7. (4 points) Consider the function

$$f(x,y) = \begin{cases} \frac{x^a y^b}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

   where a, b are nonnegative integers.

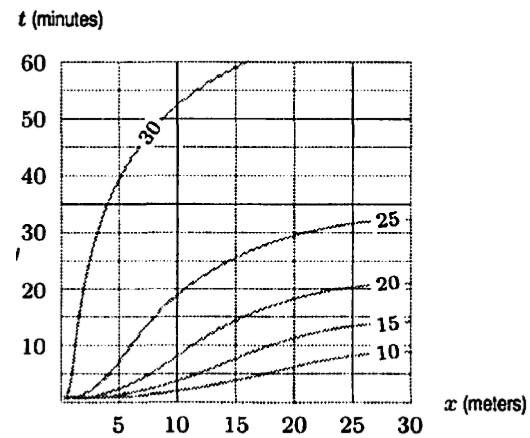
   For each of the following values of a and b, determine if the function f is continuous at $(0,0)$.

   (a) $a = 1, b = 5$

   (b) $a = 0, b = 1$

8. (4 points) The following figure shows a contour diagram for the temperature T (in Celcius) along a wall in a heated room as a function of distance x in meters along the wall and time t in minutes. Estimate $\partial T/\partial x$ and $\partial T/\partial t$ at the given points. Give the units and interpret your answers.

(a) $x = 15$, $t = 20$ (b) $x = 5$, $t = 12$



9. (4 points) A function $f(x, y, z)$ is called a harmonic function if its second-order partial derivatives exist and if it satisfies Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

(a) Is $f(x, y, z) = x^2 + y^2 - 2z^2$ harmonic? What about $f(x, y, z) = x^2 - y^2 + z^2$?

(b) We may generalize Laplace's equation to functions of n variables as:

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

Give an example of a harmonic function of 7 variables, and verify that your example is correct.

10. (4 points) A friend was asked to find the equation of the tangent plane to the surface $z = x^3 - y^2$ at the point $(x, y) = (2, 3)$. The friend's answer was

$$z = 3x^2(x - 2) - 2y(y - 3) - 1.$$

(a) At a glance, without doing <u>any</u> computation, how do you know that this is incorrect? What mistake did the friend make?

(b) Answer the question correctly.

11. (4 points) Wind chill, a measure of the apparent temperature felt on exposed skin, is a function of air temperature T and wind speed v. The following table contains the values of the wind chill $W(v, T)$ for some values of v and T.

	$T = 10$	$T = 5$	$T = 0$	$T = -10$
$v = 5$	1	-5	-11	-22
$v = 20$	-9	-15	-22	-35
$v = 25$	-11	-17	-24	-37
$v = 30$	-12	-19	-26	-39

(a) Find a linearization of the function $W(v, T)$ at the point $(v, T) = (25, 5)$.

(b) Use the above linearization to approximate $W(24, 6)$.

(c) Use the above linearization to approximate $W(5, -10)$, and explain why this value is very different from the actual value in the table above.

12. (4 points) Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the given point:

$$xe^y + ye^z + 2\ln(x) = 2 + 3\ln(2), \qquad (1, \ln(2), \ln(3)).$$

13. (4 points) If $f(u, v, w)$ is differentiable and $u = x - y, v = y - z$, and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$