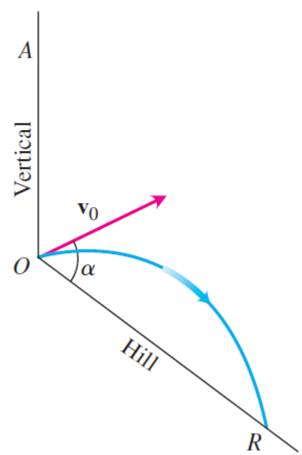
New York University
MATH.UA 123 Calculus 3

Problem Set 2

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of **??** total points.

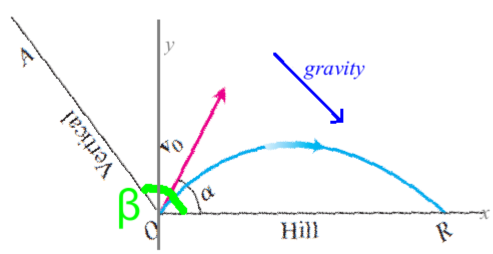
1. (4 points) A projectile is launched straight down an inclined plane as shown in he figure below. Show that the greatest downhill range (the distance from the initial position to the point where the projectile hits the ground) is achieved when the initial velocity vector bisects the angle $\angle AOR$ between the vertical line and the plane.



Solution: See figure below. Suppose that β denote the angle between the line AO and the hill.

Suppose that the initial position is at the origin. We tilt the picture so that the hill is along the x-axis. This is okay, because we will adjust the direction of gravity accordingly. Hence, the acceleration due to gravity is $\mathbf{a}(t) = \langle -g\cos\beta, -g\sin\beta \rangle$.

(Note: β is in $[\pi/2, \pi]$, so $-g\cos\beta > 0$, in agreement with our coordinate system pictured below).



Suppose that the projectile's initial speed is v_0, then the projectile's initial velocity is $\mathbf{v}_0 = \langle v_0 \cos(\alpha), v_0 \sin(\alpha)\rangle$. Then, the velocity function of the projectile is

$$\mathbf{v}(t) = \langle v_0 \cos\alpha - gt\cos\beta, v_0 \sin\alpha - gt\sin\beta\rangle.$$

Furthermore, the position function of the projectile is

$$\mathbf{r}(t) = \langle v_0 \cos(\alpha)t - 0.5g\cos(\beta)t^2, v_0\sin(\alpha)t - \frac{1}{2}g\sin(\beta)t^2\rangle.$$

Then, the projectile hits the ground when the y-component of $\mathbf{r}(t)$ is zero. That is, when

$$v_0 \sin(\alpha)t = \frac{1}{2}g\sin(\beta)t^2,$$

or, equivalently, when

$$t = \frac{2v_0}{g\sin\beta}\sin\alpha.$$

So, the downhill range as a function of the angle α is the x-component of $\mathbf{r}\left(\frac{2v_0}{g\sin\beta}\sin\alpha\right)$, which we will now call $D(\alpha)$:

$$\begin{aligned} D(\alpha) &= v_0\cos(\alpha)t - 0.5g\cos(\beta)t^2 \\ &= t(v_0\cos\alpha - 0.5g\cos(\beta)t) \\ &= \frac{2v_0}{g\sin\beta}\sin\alpha\left(v_0\cos\alpha - 0.5g\cos\beta\frac{2v_0}{g\sin\beta}\sin\alpha\right) \\ &= \frac{2v_0^2}{g\sin\beta}\left(\sin\alpha\cos\alpha - \frac{1}{\tan\beta}\sin^2\alpha\right). \end{aligned}$$

So, to maximize the downhill range, take the derivative with respect to α,

$$D'(\alpha) = \frac{2v_0^2}{g\sin\beta}\left(\cos^2\alpha - \sin^2\alpha - \frac{2\sin\alpha\cos\alpha}{\tan\beta}\right),$$

and set it to zero. Hence, $D'(\alpha) = 0$ when

$$\cos^2\alpha - \sin^2\alpha = \frac{2\sin\alpha\cos\alpha}{\tan\beta}$$

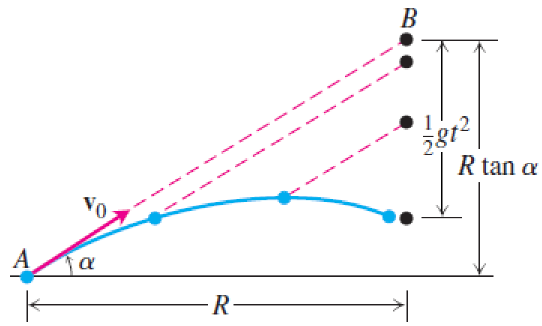
or, equivalently, when

$$\tan\beta = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \tan(2\alpha).$$

So, letting $\alpha = 0.5\beta$ will result in $D'(\alpha) = 0$. We can check that $\alpha = 0.5\beta$ is in fact where $D(\alpha)$ is maximized.

2. (4 points) The figure below shows an experiment with two marbles. Marble A was launched towards marble B with launch angle α and initial speed $v_0 = |\mathbf{v}_0| > 0$. At the same instant, marble B was released to fall from rest at $R\tan\alpha$ units directly above a spot R units horizontally downrange from A.

   Show that the marbles collide regardless of the initial speed v_0.



Solution: Since v_0 denote the initial speed of marble A, then its initial velocity is $\mathbf{v}_0 = \langle v_0 \cos\alpha, v_0 \sin\alpha\rangle$.
Let $\mathbf{r}(t)$ denote the position vector of marble A and $\mathbf{s}(t)$ denote the position vector of marble B. The acceleration vectors function of marble A and of B are the same: $\langle 0, -g\rangle$.

So, the velocity function of marble A is

$$\langle v_0 \cos\alpha, v_0 \sin\alpha - gt\rangle$$

and that of marble B is

$$\langle 0, -gt\rangle.$$

Therefore, the position vectors of A and of B are:

$$\mathbf{r}(t) = \langle v_0 \cos\alpha t, v_0 \sin\alpha t - 0.5gt^2\rangle$$

and

$$\mathbf{s}(t) = \langle R, R\tan\alpha - 0.5gt^2\rangle.$$

We will first solve for the time t at which marble A's horizontal position is R:

$$R = v_0 \cos\alpha t,$$

which means that

$$t = \frac{R}{v_0 \cos\alpha}.$$

At this time, marble A's vertical position is

$$v_0 \sin\alpha \frac{R}{v_0 \cos\alpha} - 0.5g\left(\frac{R}{v_0\cos\alpha}\right)^2 \quad = \quad R\tan\alpha - \frac{0.5gR^2}{v_0^2\cos^2\alpha}$$

At this same time, marble B's vertical position is

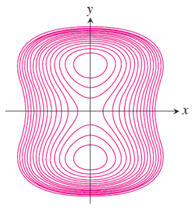
$$R\tan\alpha - 0.5g\left(\frac{R}{v_0\cos\alpha}\right)^2 \quad = \quad R\tan\alpha - \frac{0.5gR^2}{v_0^2\cos^2\alpha},$$

which is exactly the same as marble A's vertical position at this time.

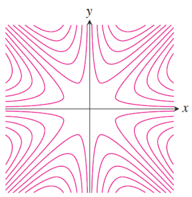
So, at time $t = \frac{R}{v_0\cos\alpha}$, marble A and B are at the same horizontal and vertical position, regardless of v_0. Thus, they always collide regardless of v_0.

3. (3 points) Match each set of level curves with the appropriate graph of function. Briefly explain your choices.

1.



2.



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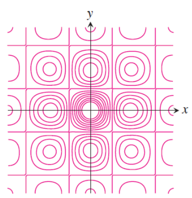

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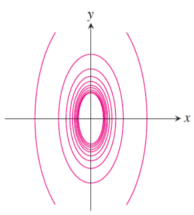

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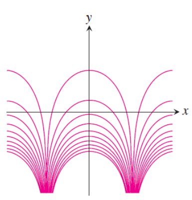

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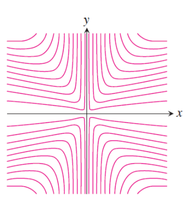
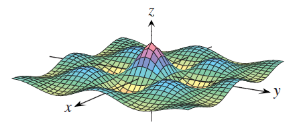
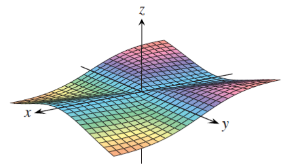


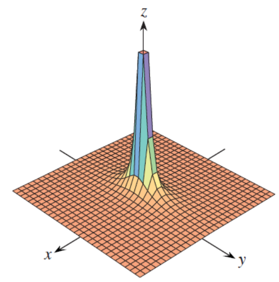
a.



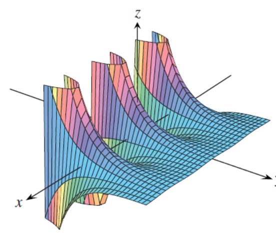
b.



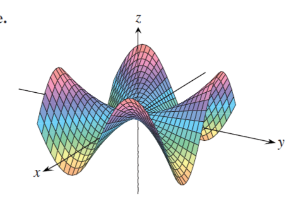
c.



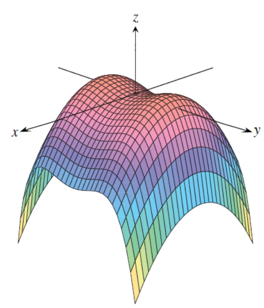
d.



e.



f.



Solution: Level curves 1 corresponds to surface f. This set of level curves indicates that there are two minima/maxima on the surface; surface f is the only one that has two peaks.

Level curves 2 correspond to surface e. This set of level curves indicates that there is an "8-fold wavy pattern", symmetric around the z-axis. This corresponds to surface e.

Level curves 3 correspond to surface a. This set of level curves indicates that there is a pattern of maxima and minimarepeated in a grid parallel to the x and y-axes, which correspond to surface a.

Level curves 4 correspond to surface c. This set of level curves indicates that the z-value increases (or decreases) quickly as we aproach the z-axis; this matches surface c.

Level curves 5 correspond to surface d. This set of level curves suggests a repeating pattern parallel to the x-axis that matches surface d.

Level curves 6 correspond to surface b. This set of level curves indicates that there is a "4-fold wavy pattern", symmetric around the z-axis. This corresponds to surface b.

4. (4 points) For each of the following functions: (i) find the function's domain, (ii) find the function's range, and (iii) sketch several of its level curves.

   (a) $f(x,y) = \frac{2y-x}{x+y+1}$

   (b) $f(x,y) = 3 - |x| - 4|y|$

   (c) $f(x,y) = \sqrt{x^2 - y^2 - 16}$

Solution:

(a) Domain: $D = \{(x,y) \in \mathbb{R}^2 \mid x+y \neq -1\}$, since the function is not defined when the denominator is zero. Geometrically, the domain of this function is the set of all points on $\mathbb{R}^2$ except those along the line $y = -1 - x$.

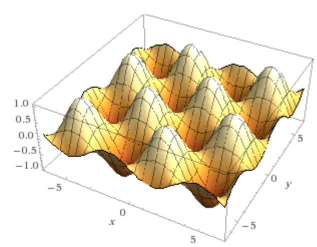
Range $= (-\infty, +\infty)$.

(b) Domain: $D = \mathbb{R}^2$. Range $= (-\infty, 3]$.

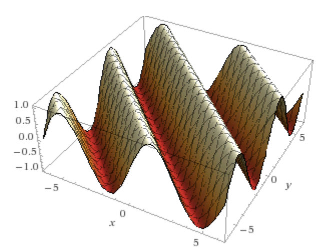
(c) Domain: $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 \geq 16\}$, the region outside the circle of radius 4, centered at the origin (the domain includes points on the circle). Range $= [0, +\infty)$.

5. (3 points) The functions $\sin(x)$ and $\cos(x)$ have wavy, periodic graphs. Manipulate these function (or devise your own) to find a function $f(x,y)$ whose graph has the following general shapes:

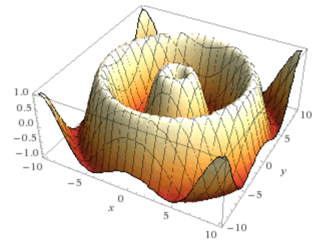
   (a) "Egg carton"



   (b) "Wavy cylinder"



   (c) "Circular wave"



Solution: Among other possible solutions:

(a) $f(x, y) = \cos(x) + \sin(y)$
Intuition: We would like a function that looks like a wave in each vertical $x = k$ or $y = k$. "Slicing" the surface above with a vertical plane $x = k$, we get a curve $z = \cos(k) + \sin(y)$, which is a sine wave (a function of y). Analogously, slicing the surface above with a vertical plane $y = k$, we get a curve $z = \cos(x) + \sin(k)$, we get a cosine wave (a function of x).

(b) $f(x, y) = \cos(x + y)$
Intuition: We would like a function whose contour plots are "diagonal" straight lines. For a fixed "height" $z = k$, the level curve of $\cos(x + y) = k$ is the straight line: $y = -x + \arccos(k) + 2\pi n$ (for integers n).

(c) $f(x, y) = \sin(x^2 + y^2)$
Intuition: We would like a function whose contour plots are circles centered at $(0, 0)$ in the xy-plane. For a fixed "height" $z = k$, the level curves of $\sin(x^2 + y^2) = k$ consists of the circles: $x^2 + y^2 = \arcsin(k) + 2\pi n$ (for integers n).

6. (4 points) (a) Use the squeeze theorem to evaluate

$$\lim_{(x,y)\to(0,0)} \tan(x)\ \sin\left(\frac{1}{|x| + |y|}\right).$$

Solution: Since the range of sine is $[-1, 1]$, we know that:

$$-\tan(x) \leq \tan(x)\ \sin\left(\frac{1}{|x| + |y|}\right) \leq \tan(x).$$

Taking the limit of both sides of each inequality:

$$\lim_{(x,y)\to(0,0)} -\tan(x) \leq \lim_{(x,y)\to(0,0)} \tan(x)\ \sin\left(\frac{1}{|x| + |y|}\right) \leq \lim_{(x,y)\to(0,0)} \tan(x).$$

Since $\lim_{(x,y)\to(0,0)} \tan(x) = 0$, then

$$0 \leq \lim_{(x,y)\to(0,0)} \tan(x)\ \sin\left(\frac{1}{|x| + |y|}\right) \leq 0.$$

By the squeeze theorem, $\lim_{(x,y)\to(0,0)} \tan(x)\ \sin\left(\frac{1}{|x| + |y|}\right) = 0.$

(b) State whether function $f(x, y) = \tan(x)\ \sin\left(\frac{1}{|x|+|y|}\right)$ is continuous at $(0, 0)$. Explain why or why not.

Solution: The function $f(x, y)$ is not continuous at $(0, 0)$. Although the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ exists (in particular, the limit is zero as we computed in part (a)), the function is not defined at $(0, 0)$. Therefore, it is not continuous at that point.

7. (4 points) Consider the function

$$f(x,y) = \begin{cases} \frac{x^a y^b}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

where a, b are nonnegative integers.

For each of the following values of a and b, determine if the function f is continuous at $(0, 0)$.

(a) $a = 1, b = 5$

Solution: Then, the function that we're working with is

$$f(x,y) = \begin{cases} \frac{xy^5}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Note that $\frac{xy^5}{x^2+y^2} = \frac{y^2}{x^2+y^2}\, xy^3$. Since $0 \leq \frac{y^2}{x^2+y^2} \leq 1$, then

$$0 \times xy^3 \leq \frac{xy^5}{x^2+y^2} \leq 1 \times xy^3$$

$$0 \leq \frac{xy^5}{x^2+y^2} \leq xy^3$$

Taking the limit as (x, y) approaches $(0, 0)$ of all sides of the inequalities:

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{xy^5}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} xy^3,$$

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{xy^5}{x^2+y^2} \leq 0.$$

By the squeeze theorem, the limit of $\frac{xy^5}{x^2+y^2}$ as (x, y) approaches $(0, 0)$ is zero, which is equal to the value of $f(x, y)$ at $(0, 0)$. Therefore, f is continuous at $(0, 0)$.

(b) $a = 0, b = 1$

Solution: Then, the function that we're working with is

$$f(x,y) = \begin{cases} \frac{y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Note that we cannot use squeeze theorem as we did in part (a). In fact, we can show that f goes to infinity as (x, y) approaches $(0, 0)$ along the y-axis.

Consider points along the y-axis, approaching $(0, 0)$: $(0, y)$ with y approaching 0. Then,

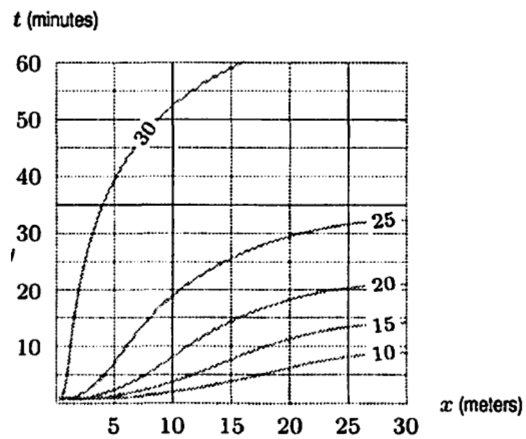
$$\lim_{(0,y)\to(0,0)} \frac{y}{x^2+y^2} = \lim_{y\to 0} \frac{y}{y^2} = \lim_{y\to 0} \frac{1}{y} = +\infty.$$

Therefore, $f(x,y)$ is not continuous at $(0,0)$.

8. (4 points) The following figure shows a contour diagram for the temperature T (in Celcius) along a wall in a heated room as a function of distance x in meters along the wall and time t in minutes. Estimate $\partial T/\partial x$ and $\partial T/\partial t$ at the given points. Give the units and interpret your answers.

(a) $x = 15$, $t = 20$

(b) $x = 5$, $t = 12$



Solution:

(a) We can estimate $\frac{\partial T}{\partial t}$ at the point $(x,y) = (15,20)$ by considering the change in temperature between points $(15,15)$ and $(15,25)$ (that is, holding x constant at 15 while varying the time t). Since $T(15,15) \approx 20$ and $T(15,25) \approx 25$, then

$$\frac{\partial T}{\partial t}(15,20) \approx \frac{\Delta T}{\Delta t} \approx \frac{25-20}{25-15} = 0.5.$$

Similarly, by considering the change in temperature from point $(10,20)$ to point $(25,20)$ (holding t constant at 20), we get a change of temperature from 25 to 20. So,

$$\frac{\partial T}{\partial x}(15,20) \approx \frac{\Delta T}{\Delta x} \approx \frac{20-25}{25-10} = -\frac{1}{3}.$$

(b) We can estimate $\frac{\partial T}{\partial t}$ at the point $(x,y) = (5,12)$ by considering the change in temperature between points $(5,7)$ and $(5,40)$ (that is, holding x constant at 5 while varying the time t). Since $T(5,7) \approx 25$ and $T(5,40) \approx 30$, then

$$\frac{\partial T}{\partial t}(5,12) \approx \frac{\Delta T}{\Delta t} \approx \frac{30-25}{40-7} = \frac{5}{33}.$$

Similarly, by considering the change in temperature from point $(2,12)$ to point $(7,12)$ (holding t constant at 12), we get a change of temperature from 2 to 7. So,

$$\frac{\partial T}{\partial x}(15,20) \approx \frac{\Delta T}{\Delta x} \approx \frac{25-30}{7-2} = -1.$$

9. (4 points) A function $f(x,y,z)$ is called a harmonic function if its second-order partial derivatives exist

and if it satisfies Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

(a) Is $f(x, y, z) = x^2 + y^2 - 2z^2$ harmonic? What about $f(x, y, z) = x^2 - y^2 + z^2$?

(b) We may generalize Laplace's equation to functions of n variables as:

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

Give an example of a harmonic function of 7 variables, and verify that your example is correct.

Solution:

(a) For $f(x, y, z) = x^2 + y^2 - 2z^2$,

$$f_{xx}(x, y, z) = 2, \; f_{yy}(x, y, z) = 2, \; f_{zz}(x, y, z) = -4.$$

So, $f_{xx} + fyy + f_{zz} = 2 + 2 + (-4) = 0$, which shows that f is harmonic.
For the function $f(x, y, z) = x^2 - y^2 + z^2$,

$$f_{xx}(x, y, z) = 2, \; f_{yy}(x, y, z) = -2, \; f_{zz}(x, y, z) = 2.$$

So, $f_{xx} + fyy + f_{zz} = 2 + (-2) + 2 = 2 \neq 0$, which shows that f is not harmonic.

(b) One example: $f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_6^2 - 6x_7^2$. Then, for each variable x_i where $1 \leq i \leq n - 1$, we can show that $f_{x_i x_i} = 2$. Furthermore, $f_{x_7 x_7} = -12$. So,

$$f_{x_1 x_1} + \ldots + f_{x_6 x_6} + f_{x_7 x_7} = 2 \times 6 - 12 = 0,$$

which shows that f is a harmonic function.

(Grading note: any example involving 3 variables or more that satisfies Laplace's equation is acceptable. The solution must verify that Laplace's equation is satisfied.)

10. (4 points) A friend was asked to find the equation of the tangent plane to the surface $z = x^3 - y^2$ at the point $(x, y) = (2, 3)$. The friend's answer was

$$z = 3x^2(x - 2) - 2y(y - 3) - 1.$$

(a) At a glance, without doing <u>any</u> computation, how do you know that this is incorrect? What mistake did the friend make?

Solution: This equation is not a linear equation!

(b) Answer the question correctly.

Solution: The equation is $z = 12(x - 2) - 6(y - 3) - 1$ (i.e. evaluate the partial derivatives at $(x, y) = (2, 3)$).

11. (4 points) Wind chill, a measure of the apparent temperature felt on exposed skin, is a function of air temperature T and wind speed v. The following table contains the values of the wind chill $W(v, T)$ for

some values of v and T.

	$T = 10$	$T = 5$	$T = 0$	$T = -10$
$v = 5$	1	-5	-11	-22
$v = 20$	-9	-15	-22	-35
$v = 25$	-11	-17	-24	-37
$v = 30$	-12	-19	-26	-39

(a) Find a linearization of the function $W(v, T)$ at the point $(v, T) = (25, 5)$.

Solution: The linearization:

$$L(v, T) = W(25, 5) + \frac{\partial W}{\partial v}(25, 5)\ (v - 25) + \frac{\partial W}{\partial T}(25, 5)\ (t - 5).$$

We use the table to approximate $\frac{\partial W}{\partial v}(25, 5)$ and $\frac{\partial W}{\partial T}(25, 5)$:

$$\begin{aligned} \frac{\partial W}{\partial v}(25, 5) &\approx \frac{W(25, 5) - W(30, 5)}{25 - 30} = -\frac{2}{5}, \\ \frac{\partial W}{\partial T}(25, 5) &\approx \frac{W(25, 5) - W(25, 10)}{5 - 10} = \frac{6}{5}. \end{aligned}$$

Therefore, the linearization is

$$L(v, T) = -17 - 0.4(v - 25) + 1.2\ (t - 5).$$

(b) Use the above linearization to approximate $W(24, 6)$.

Solution:

$$W(24, 6) \approx L(24, 6) = -17 - 0.4 \times (-1) + 1.2 \times 1 = -15.4$$

(c) Use the above linearization to approximate $W(5, -10)$, and explain why this value is very different from the actual value in the table above.

Solution:

$$W(5, -10) \approx L(5, -10) = -17 - 0.4 \times (-20) + 1.2 \times (-15) = -27.$$

There is a difference of -5 units between this and the actual value of $W(5, -10) = -22$. The estimation of $W(5, -10)$ using the linearization at $(25, 5)$ is not accurate because the point $(5, -10)$ is not close to the point $(25, 5)$. The linearization is only accurate for points close to the point at which the linearization is computed.

12. (4 points) Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the given point:

$$xe^y + ye^z + 2\ln(x) = 2 + 3\ln(2), \qquad (1, \ln(2), \ln(3)).$$

Solution: We use implicit differentiation. First, find $\frac{\partial z}{\partial x}$:

$$e^y + ye^z \frac{\partial z}{\partial x} + \frac{2}{x} = 0,$$

$$\frac{\partial z}{\partial x} = -\frac{e^y + 2/x}{ye^z}.$$

Evaluate at the point $(1, \ln(2), \ln(3))$:

$$\frac{\partial z}{\partial x} = -\frac{e^{\ln(2)} + 2}{\ln(2)e^{\ln(3)}} = -\frac{4}{3\ln(3)}.$$

Next, find $\frac{\partial z}{\partial y}$:

$$xe^y + e^z + ye^z \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial z}{\partial y} = -\frac{xe^y + e^z}{ye^z}.$$

Evaluate at $(1, \ln(2), \ln(3))$:

$$\frac{\partial z}{\partial y} = -\frac{e^{\ln(2)} + e^{\ln(3)}}{\ln(2)e^{\ln(3)}} = -\frac{5}{3\ln(3)}.$$

13. (4 points) If $f(u, v, w)$ is differentiable and $u = x - y, v = y - z$, and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\
&= \frac{\partial f}{\partial u} + 0 - \frac{\partial f}{\partial w} \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \\
&= -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} + 0 \\
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} \\
&= 0 - \frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \\
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} &= \left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}\right) + \left(-\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + \left(-\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\right) \\
&= 0.
\end{aligned}$$