New York University
MATH.UA 123 Calculus 3

Problem Set 3

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (7 points) Consider the function $f(x, y) = kx^2 + y^2 - 4xy$, where k is some fixed constant.
   (a) Show that for any value of k, $(0, 0)$ is a critical point of f.
   (b) Determine the values of k (if any) for which $(0, 0)$ is
      (a) a saddle point,
      (b) a local maximum,
      (c) a local minimum.

2. (7 points) We want to find the absolute maximum value of

$$f(x, y) = x^3 - 3x - y^2 + 12$$

   on the closed square $D = \{(x, y) \mid -2 \le x \le 3,\ -2 \le y \le 3\}$.
   (a) First, find the critical points of f, and determine if any of the critical points is a local maximum.
   (b) Then, find the (absolute) maximum value of f along the boundary of D: the bottom edge of D and along the top arc of D. (Note: some of the values might be a bit messy. Give solutions to two decimal places.)
   (c) Using parts (a) and (b), find the absolute maximum value of f on D.

3. (7 points) [1] **Method of Least Squares**
   Consider the points $(0, 1)$, $(1, 0)$, $(2, 2)$. We would like to try and find a line of the form $f(x) = mx + b$ which "fits" the data as well as possible. One method to do this is the following: for each of the three points (x, y) above compute the value

$$d(x, y) = y - (mx + b).$$

   For example:

$$d(0, 1) = 1 - (m \cdot 0 + b) = 1 - b.$$

   Note that $|d|$ measures the distance between the actual value of y and the one suggested by the line $f(x) = mx + b$.

   After we compute the values of each d for the three points we would like to solve the following: Find m and b which minimizes

$$|d(0, 1)| + |d(1, 0)| + |d(2, 2)|.$$

   Since absolute values are not differentiable everywhere, we replace this condition with:

Find m and b which minimizes $$(d(0, 1))^2 + (d(1, 0))^2 + (d(2, 2))^2.$$

[1] Challenge problem

The solution for m and b above will give us a line

$$f(x) = mx + b$$

which we call the least squares approximation for the data.

(a) Solve the minimization problem above (boxed). Plot the three data points and the line that you compute in your solution.

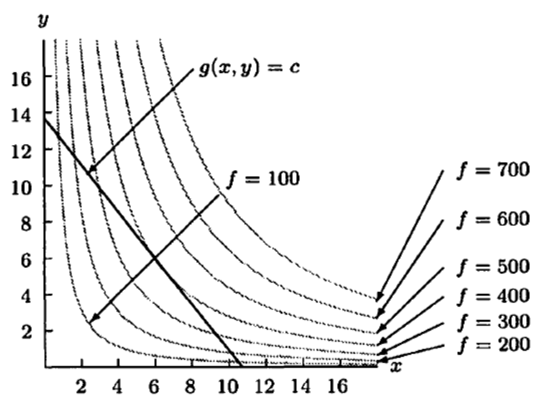
(b) Replace the points with three general data points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3)$$

and define $d_1 = d(x_1, y_1)$ etc. State the minimization problem you would need to solve to find the least squares line. What system of equations do you derive by taking partial derivatives with respect to m and b?

(c) Generalize (b) to n general data points.

4. (7 points) The figure below shows contours of f, a function of x and y, and the line that satisfies a constraint $g(x, y) = c$.



(a) Does f have a maximum value subject to the constraint $g(x, y) = c$ for $x \geq 0$, $y \geq 0$? If so, approximate where it is and what its value is. Show all work/justification.

(b) Does f have a minimum value subject to the constraint $g(x, y) = c$ for $x \geq 0$, $y \geq 0$? If so, approximate where it is and what its value is. Show all work/justification.

5. (8 points) Find the maximum value that $f(x, y, z) = x^2 + 2y - z^2$ can have on the line of intersection of the planes $2x - y = 0$ and $y + z = 0$.

6. (7 points) Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a type of curve known as a "piriform").

(a) Try using Lagrange multipliers to solve the problem.

(b) Show that the minimum value is $f(0, 0)$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value λ.

(c) Explain why Lagrange multipliers fail to find the minimum value in this case. (Hint for parts (b) and (c): try to plot the curve using Wolfram Alpha, and locate where the point $(0, 0)$ is on this curve.)

7. (7 points) Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.