New York University
MATH.UA 123 Calculus 3

Problem Set 3

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (7 points) Consider the function $f(x,y) = kx^2 + y^2 - 4xy$, where k is some fixed constant.
   (a) Show that for any value of k, $(0,0)$ is a critical point of f.

   > **Solution:** $f_x(x,y) = 2kx - 4y$ and $f_y(x,y) = 2y - 4x$. Since $f_x(0,0) = 0$ and $f_y(0,0) = 0$, then $(0,0)$ is a critical point of f.

   (b) Determine the values of k (if any) for which $(0,0)$ is
      (a) a saddle point,
      (b) a local maximum,
      (c) a local minimum.

   > **Solution:** $f_{xx}(x,y) = 2k$, $f_{xy}(x,y) = f_{yx}(x,y) = -4$, and $f_{yy}(x,y) = 2$. Therefore, $D(0,0) = 4k - 16$.
   > So, $(0,0)$ is a saddle point when $4k - 16 < 0$, or equivalently, when $k < 4$. $D(0,0) > 0$ when $k > 4$; since $f_{xx} > 4$ for $k > 4$, then $(0,0)$ is a local minimum for $k > 4$. $(0,0)$ is never a local maximum for any value of k.

2. (7 points) We want to find the absolute maximum value of

$$f(x,y) = x^3 - 3x - y^2 + 12$$

   on the closed square $D = \{(x,y) \mid -2 \leq x \leq 3,\ -2 \leq y \leq 3\}$.
   (a) First, find the critical points of f, and determine if any of the critical points is a local maximum.

   > **Solution:** Since $f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -2y$, then the critical points of f are $(-1,0), (1,0)$.
   > We use the second derivative test. Note that $f_{xx}(x,y) = 6x, f_{yy}(x,y) = -2, f_{xy}(x,y) = 0$. Therefore, $D(x,y) = -12x$. Since, $D(-1,0) > 0$ and $f_{xx}(-1,0) < 0$, then $(-1,0)$ is a local maximum, with $f(-1,0) = 2$. (Similarly, we can show that $(1,6)$ is a local minimum.)

   (b) Then, find the (absolute) maximum value of f along the boundary of D: the bottom edge of D and along the top arc of D. (Note: some of the values might be a bit messy. Give solutions to two decimal places.)

Solution: The bottom edge of D is the line segment between $(0,-3)$ and $(0,3)$. This line segment contains points $(x,0)$ with $-3 \le x \le 3$. Then, along this line segment the function f is just

$$f(x,0) = x^3 - 3x.$$

Call $g(x) = f(x,0) = x^3 - 3x$. The critical points of g are:

$$0 = g'(x) = 3x^2 - 3$$

$$x = \pm 1.$$

Check the value of f at the critical points $(\pm 1, 0)$ and at the endpoints of the line segment, $(\pm 3, 0)$:

$$f(-1,0) = 2,\ f(1,0) = -2,\ f(-3,0) = -18,\ f(3,0) = 18.$$

So, along the bottom edge of D, the maximum is attained at $(3,0)$, with $f(3,0) = 18$.
Next, we consider the top arc: $x^2 + y^2 = 9$, $y \ge 0$. Note that points on this arc satisfies: $(x,y) = (x, \sqrt{9-x^2})$, with $-3 \le x \le 3$. Then, along this arc, the function f is just

$$h(x) := f(x, \sqrt{9-x^2}) = x^3 - 3x - (9 - x^2) + 12.$$

Finding the critical points of h:

$$0 = h'(x) = 3x^2 - 3 + 2x,$$

and using the quadratic formula:

$$x = \frac{-2 \pm \sqrt{4+36}}{6} = -\frac{1}{3} \pm \frac{\sqrt{10}}{3} \approx -1.39, 0.72.$$

Computing the corresponding y values, the critical points are at $\approx (-1.39, 2.66),\ (0.72, 2.91)$.
Check the value of f at the critical points $(\pm 1, 0)$ and at the endpoints of the line segment, $(\pm 3, 0)$:

$$f(-1.39, 2.66) \approx 6.41,\ f(0.72, 2.91) \approx 1.73,\ f(-3,0) = -18,\ f(3,0) = 18.$$

So, along the top arc of D, the maximum is attained at $(3,0)$, with $f(3,0) = 18$.

(c) Using parts (a) and (b), find the absolute maximum value of f on D.

Solution: The max value of f is $f(0,3) = 18$, attained on the boundary of D.

3. (7 points) [1] **Method of Least Squares**
Consider the points $(0,1)$, $(1,0)$, $(2,2)$. We would like to try and find a line of the form $f(x) = mx + b$ which "fits" the data as well as possible. One method to do this is the following: for each of the three points (x,y) above compute the value

$$d(x,y) = y - (mx + b).$$

For example:

$$d(0,1) = 1 - (m \cdot 0 + b) = 1 - b.$$

Note that $|d|$ measures the distance between the actual value of y and the one suggested by the line $f(x) = mx + b$.

[1] Challenge problem

After we compute the values of each d for the three points we would like to solve the following: Find m and b which minimizes

$$|d(0,1)| + |d(1,0)| + |d(2,2)|.$$

Since absolute values are not differentiable everywhere, we replace this condition with:

Find m and b which minimizes

$$(d(0,1))^2 + (d(1,0))^2 + (d(2,2))^2.$$

The solution for m and b above will give us a line

$$f(x) = mx + b$$

which we call the least squares approximation for the data.

(a) Solve the minimization problem above (boxed). Plot the three data points and the line that you compute in your solution.

Solution:

$$\begin{aligned} f(m,b) &= (1-b)^2 + (0-mx-b)^2 + (2-2m-b)^2 \\ &= 5 + 5m^2 + 3b^2 - 6b - 8m + 6mb. \\ f_m(m,b) &= 10m + 6b - 8 \\ f_b(m,b) &= 6m + 6b - 6. \end{aligned}$$

So, $(0.5, 0.5)$ is a critical point. Furthermore, $D(m,b) = 10\times 6 - 6^2 > 0$ and $f_{mm}(m,b) = 10 > 0$. Therefore, it is a local minimum. Since the function is concave up everywhere, it is also an absolute minimum

(b) Replace the points with three general data points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3)$$

and define $d_1 = d(x_1, y_1)$ etc. State the minimization problem you would need to solve to find the least squares line. What system of equations do you derive by taking partial derivatives with respect to m and b?

Solution: The minimization problem: find the minimum of $f(m,b)$ given below:

$$\begin{aligned} f(m,b) &= (y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + (y_3 - mx_3 - b)^2 \\ &= (\sum_{i=1}^{3} y_i^2) + m^2(\sum_{i=1}^{3} x_i^2) + 3b^2 - 2m(\sum_{i=1}^{3} x_i y_i) + 2mb(\sum_{i=1}^{3} x_i) - 2b(\sum_{i=1}^{3} y_i). \end{aligned}$$

The system of equations we derive to solve for the minimum of f:

$$\begin{aligned} 0 &= 2m(\sum_{i=1}^{3} x_i^2) + 2b(\sum_{i=1}^{3} x_i) - 2(\sum_{i=1}^{3} x_i y_i) \\ 0 &= 2m(\sum_{i=1}^{3} x_i) + 6b - 2(\sum_{i=1}^{3} y_i). \end{aligned}$$

(Stopping here is fine since the problem only asks for the system of equations, but if we want to actually solve for m and b, continue below.)

Multiply the first equation by 3 and the second by $(\sum_{i=1}^{3} x_i)$:

$$\begin{aligned}
0 &= 6m(\sum_{i=1}^{3} x_i^2) + 6b(\sum_{i=1}^{3} x_i) - 6(\sum_{i=1}^{3} x_i y_i) \\
0 &= 2m(\sum_{i=1}^{3} x_i)(\sum_{i=1}^{3} x_i) + 6b(\sum_{i=1}^{3} x_i) - 2(\sum_{i=1}^{3} y_i)(\sum_{i=1}^{3} x_i),
\end{aligned}$$

and subtract the second from the first:

$$0 = 2m\left(3(\sum_{i=1}^{3} x_i^2) - (\sum_{i=1}^{3} x_i)^2\right) - 2\left(3(\sum_{i=1}^{3} x_i y_i) - (\sum_{i=1}^{3} y_i)(\sum_{i=1}^{3} x_i)\right).$$

Solving for m:

$$m = \frac{3(\sum_{i=1}^{3} x_i y_i) - (\sum_{i=1}^{3} y_i)(\sum_{i=1}^{3} x_i)}{3(\sum_{i=1}^{3} x_i^2) - (\sum_{i=1}^{3} x_i)^2}.$$

Solving for b (using the second of the inital system of equations):

$$b = \frac{(\sum_{i=1}^{3} y_i)}{3} - m\frac{(\sum_{i=1}^{3} x_i)}{3}$$

or:

$$b = \frac{(\sum_{i=1}^{3} y_i)}{3} - \frac{(\sum_{i=1}^{3} x_i y_i) - (\sum_{i=1}^{3} y_i)(\sum_{i=1}^{3} x_i)}{3 - (\sum_{i=1}^{3} x_i)}.$$

(c) Generalize (b) to n general data points.

Solution: The minimization problem: find the minimum of $f(m, b)$ given below:

$$\begin{aligned}
f(m,b) &= (y_1 - mx_1 - b)^2 + \ldots + (y_n - mx_n - b)^2 \\
&= (\sum_{i=1}^{n} y_i^2) + m^2(\sum_{i=1}^{n} x_i^2) + nb^2 - 2m(\sum_{i=1}^{n} x_i y_i) + 2mb(\sum_{i=1}^{n} x_i) - 2b(\sum_{i=1}^{n} y_i).
\end{aligned}$$

The system of equations we derive to solve for the minimum of f:

$$\begin{aligned}
0 &= 2m(\sum_{i=1}^{n} x_i^2) + 2b(\sum_{i=1}^{n} x_i) - 2(\sum_{i=1}^{n} x_i y_i) \\
0 &= 2m(\sum_{i=1}^{n} x_i) + 2nb - 2(\sum_{i=1}^{n} y_i).
\end{aligned}$$

(Stopping here is fine since the problem only asks for the system of equations, but if we want to actually solve for m and b, continue below.)
Multiply the first equation by n and the second by $(\sum_{i=1}^{n} x_i)$:

$$\begin{aligned}
0 &= 2nm(\sum_{i=1}^{n} x_i^2) + 2nb(\sum_{i=1}^{n} x_i) - 2n(\sum_{i=1}^{n} x_i y_i) \\
0 &= 2m(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} x_i) + 2nb(\sum_{i=1}^{n} x_i) - 2(\sum_{i=1}^{n} y_i)(\sum_{i=1}^{n} x_i),
\end{aligned}$$

and subtract the second from the first:

$$0 = 2m\left(n(\sum_{i=1}^{n} x_i^2) - (\sum_{i=1}^{n} x_i)^2\right) - 2\left(n(\sum_{i=1}^{n} x_i y_i) - (\sum_{i=1}^{n} y_i)(\sum_{i=1}^{n} x_i)\right).$$

Solving for m:

$$m = \frac{n(\sum_{i=1}^{n} x_i y_i) - (\sum_{i=1}^{n} y_i)(\sum_{i=1}^{n} x_i)}{n(\sum_{i=1}^{n} x_i^2) - (\sum_{i=1}^{n} x_i)^2}.$$

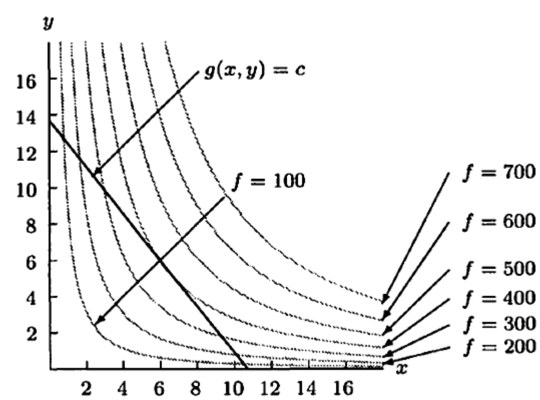
Solving for b (using the second of the inital system of equations):

$$b = \frac{(\sum_{i=1}^{n} y_i)}{n} - m\frac{(\sum_{i=1}^{n} x_i)}{n}$$

or:

$$b = \frac{(\sum_{i=1}^{n} y_i)}{n} - \frac{(\sum_{i=1}^{3} x_i y_i) - (\sum_{i=1}^{n} y_i)(\sum_{i=1}^{n} x_i)}{n - (\sum_{i=1}^{n} x_i)}.$$

4. (7 points) The figure below shows contours of f, a function of x and y, and the line that satisfies a constraint $g(x, y) = c$.



(a) Does f have a maximum value subject to the constraint $g(x, y) = c$ for $x \geq 0$, $y \geq 0$? If so, approximate where it is and what its value is. Show all work/justification.

(b) Does f have a minimum value subject to the constraint $g(x, y) = c$ for $x \geq 0$, $y \geq 0$? If so, approximate where it is and what its value is. Show all work/justification.

Solution:

(a) Yes. We can use the method of Lagrange multipliers: Find the point along $g = c$ at which the gradient of g and the gradient of f are scalar multiples of one another. This point is approximately $(6, 6)$, with $f(6, 6) = 300$. Furthermore, no other points in $g = c$ intersects level curves of f of level higher than 300.

(b) In part (a), the only solution to the Lagrange multipliers method is the point $(6, 6)$, and this is not where f is minimized. So, we need to check the endpoints of the line $g = c$. As we go towards the endpoints, which is where $g = c$ meets the x-axis (i.e., $y = 0$) and where it meets the y-axis (i.e., $x = 0$), the value of f keeps decreasing. Thus, the minimum value must happen at one of the endoints: at around $(11, 0)$ or at around $(0, 13.8)$ (or both).

5. (8 points) Find the maximum value that $f(x,y,z) = x^2 + 2y - z^2$ can have on the line of intersection of the planes $2x - y = 0$ and $y + z = 0$.

Solution: Let $g(x,y,z) = 2x - y$ and $h(x,y,z) = y + z$. Solve for λ, μ, x, y, z that satisfy:

$$2x = 2\lambda \tag{1}$$
$$2 = -\lambda + \mu \tag{2}$$
$$-2z = \mu \tag{3}$$
$$2x - y = 0 \tag{4}$$
$$y + z = 0 \tag{5}$$

From (6) and (9), we see that

$$x = \lambda \tag{1'}$$
$$x = \frac{y}{2}. \tag{4'}$$

Furthermore, subtracting (8) from (7), we obtain

$$2 + 2z = -\lambda = -x = -y/2 \tag{2'}$$

From (2') and (5), we have two equations in two unknowns:

$$2 + 2z + y/2 = 0 \tag{2''}$$
$$y + z = 0. \tag{5'}$$

Solving for y and z, we obtain: $y = 4/3$ and $z = -4/3$.

From(1') and (4'), since $\lambda = x = y/2 = 2/3$. From (8), $\mu = -2z = 8/3$.

So, the only critical point is: $(x,y,z) = (2/3, 4/3, -4/3)$ with $\mu = 8/3$ and $\lambda = 2/3$, with function value is: $f(2/3, 4/3, -4/3) = \frac{4}{9} + \frac{8}{3} - \frac{16}{9} = \frac{4}{3}$.

This is the maximum, since other points on the line of intersection give a smaller value of f (for example, the point $(0,0,0)$ is on the line of intersection, and $f(0,0,0) = 0$).

6. (7 points) Consider the problem of minimizing the function $f(x,y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a type of curve known as a "piriform").

   (a) Try using Lagrange multipliers to solve the problem.

Solution: The system of equations is:

$$\begin{aligned} 1 &= \lambda(4x^3 - 3x^2) \\ 0 &= 2\lambda y \\ y^2 + x^4 - x^3 &= 0 \end{aligned}$$

The first constraint implies that $\lambda \neq 0$ and $x \neq 0$. So, in the second constraint, it must be the case that $y = 0$.

Since $y = 0$, then we can use the third constraint to solve for x: $x^4 - x^3 = 0$, which means that $x = 0$ or $x = 1$. Since x cannot equal 0, then the only solution for x is $x = 1$.

So, the only solution is $(1,0)$, with value $f(1,0) = 1$. However, $(1,0)$ cannot be the point that minimizes f. For instance, the point $(0,0)$ satisfies the constraint and $f(0,0) = 0$, which is smaller.

(b) Show that the minimum value is $f(0,0)$ but the Lagrange condition $\nabla f(0,0) = \lambda \nabla g(0,0)$ is not satisfied for any value λ.

Solution: To show that $f(0,0) = 0$ is the minimum, we must show that if $x < 0$, then there is no y that satisfies $y^2 + x^4 - x^3 = 0$:

If $x < 0$, then $x^3 < 0$ and $x^4 > 0$. So, $y^2 = x^3 - x^4 < 0$. So, y has no real-valued solution.

So, $f(x,y) = x = 0$ is the smallest possible value of f in the region specified by the constraint.

(c) Explain why Lagrange multipliers fail to find the minimum value in this case. (Hint for parts (b) and (c): try to plot the curve using Wolfram Alpha, and locate where the point $(0,0)$ is on this curve.)

Solution: The method of Lagrange multipliers solves for the point (x,y) at which the tangent plane to the curve $g(x,y) = c$ is parallel to the tangent curve to the level curve $f(x,y) = k$. However, at the point $(0,0)$, the curve $g(x,y) = c$ is "pointy" (not smooth) and does not have a well-defined tangent plane.

7. (7 points) Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

Solution: Problem formulation: We can assume that the sides of the box are parallel to the coordinate planes. A box inside the unit sphere that has the largest possible volume must touch the sphere at its corner points.

Thus, we can assume that the eight corner points of the box are $(\pm a, \pm b, \pm c)$, where $a^2 + b^2 + c^2 = 1$ (since the corner points must lie on the unit sphere). For simplicity, we can also assume that $a, b, c \geq 0$.

So, the dimension of the box is $(2a) \times (2b) \times (2c)$ and the volume is $f(a,b,c) = 8abc$.

So, we want to maximize $f(a,b,c) = 8abc$ subject to the constraint $a^2 + b^2 + c^2 = 1$.

Solve using the method of Lagrange multipliers. We want to find λ, a, b, c that satisfies:

$$8bc = 2a\lambda \tag{6}$$
$$8ac = 2b\lambda \tag{7}$$
$$8ab = 2c\lambda \tag{8}$$
$$a^2 + b^2 + c^2 = 1. \tag{9}$$

Solving for a in equation (6), we get

$$a = \frac{4bc}{\lambda}. \tag{1'}$$

Substituting this for a in equations (7) and (8), we get

$$\frac{16bc^2}{\lambda} = b\lambda,$$
$$\frac{16b^2c}{\lambda} = c\lambda.$$

which can be simplified to

$$16c^2 = \lambda^2, \tag{2'}$$
$$16b^2 = \lambda^2. \tag{3'}$$

Substituting (1'), (2'), and (3') into (9), we get:

$$\frac{16b^2c^2}{\lambda^2} + \frac{\lambda^2}{16} + \frac{\lambda^2}{16} = 1.$$

Using (2') and (3') to replace the first term, we get

$$\frac{\lambda^2}{16} + \frac{\lambda^2}{16} + \frac{\lambda^2}{16} = 1.$$

Therefore, $\lambda = \frac{\pm 4}{\sqrt{3}}$. Using (1'), (2'), and (3') again, and keeping in mind that we are solving for $a, b, c > 0$:

$$a = b = c = \frac{1}{\sqrt{3}}.$$

Therefore, the dimension of the largest box inside the unit sphere is $\frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}$, a cube. Its volume is $\frac{8}{3\sqrt{3}}$.