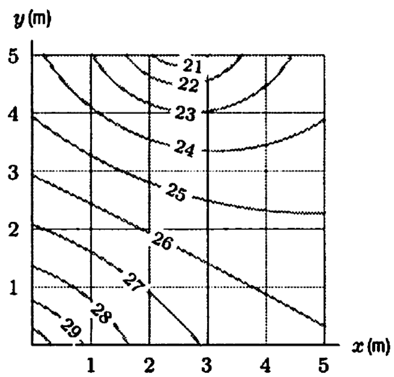
New York University
MATH.UA 123 Calculus 3

Problem Set 4

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

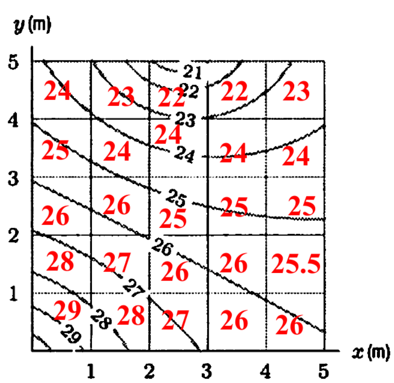
1. (5 points) The figure below shows the distribution of temperature, in °C, in a 5 meter by 5 meter heated room. Using Riemann sums, estimate the average temperature in the room.



Solution: Call the region (the room) R, and let $T(x, y)$ denote the temperature at point (x, y) in R. Then, the average temperature is:

$$\frac{1}{\text{Area of room}} \iint_R T(x, y)\; dA.$$

The total area of the room is 25 square meter. To estimate the above integral, divide the room into 25 squares, each with dimension 1 meter by 1 meter. So, the area of each small square is 1 square meter. Choose one sample point from each square and find the temperature at this chosen point. One possibility is as follows:



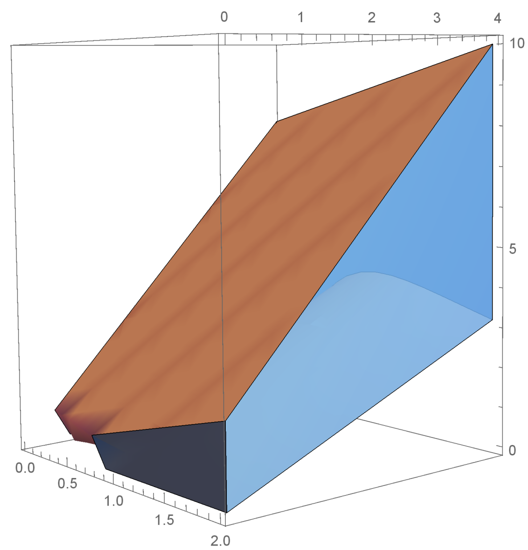
Then, $\iint_R T(x,y)\, dA$ is approximately the sum of the above temperatures times 1 square meter, which is

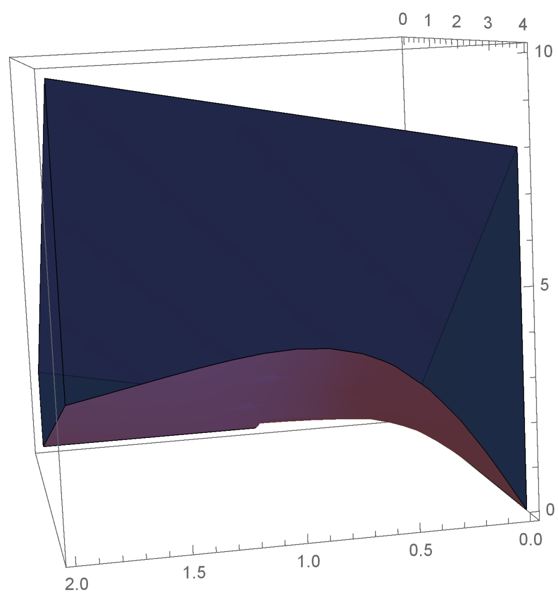
$$24 + 23 + 22 + \ldots 27 + 26 + 26 = 630.5.$$

So, the average temperature is approximately $\frac{630.5}{25} = 25.22$.

2. (5 points) Sketch the solid that lies between the surface $z = \frac{2xy}{x^2+1}$ and the plane $z = x + 2y$ and is bounded by the planes $x = 0$, $x = 2$, $y = 0$, and $y = 4$. Then, find its volume.

Solution: The graphs below are the solid from different angle.





The plane $z = x + 2y$ is above the surface $z = \frac{2xy}{x^2+1}$. Thus, the volume is

$$\begin{aligned}
\int_0^2 \int_0^4 (x + 2y - \frac{2xy}{x^2+1})dydx &= \int_0^2 \left[xy + y^2 - \frac{xy^2}{x^2+1}\right]_0^4 dx \\
&= \int_0^2 (4x + 16 - \frac{16x}{x^2+1})dx \\
&= \int_0^2 (4x+16)dx - \int_0^2 \frac{16x}{x^2+1}dx \qquad (*) \\
&= \left[2x^2 + 16x\right]_0^2 - \left[8\ln\left(x^2+1\right)\right]_0^2 \\
&= 40 - 8\ln 5
\end{aligned}$$

(*) is because

$$\int_0^2 \frac{16x}{x^2+1}dx = \int \frac{8}{u}du = \left[8\ln\left(x^2+1\right)\right]_0^2 = 8\ln 5$$

3. (5 points) Evaluate the double integral

$$\int\int_R \frac{y}{x^2y^2+1}\, dA,$$

over the region R: $0 \leq x \leq 1$, $-1 \leq y \leq 2$.

Solution:

$$\begin{aligned}
\int_{-1}^2 \int_0^1 \frac{y}{x^2y^2+1}dxdy &= \int_{-1}^2 \left[\arctan xy\right]_0^1 dy \qquad (*) \\
&= \int_{-1}^2 \arctan y dy \\
&= \left[y\arctan y - \frac{1}{2}\ln\left(1+y^2\right)\right]_{-1}^2 \qquad (**) \\
&= 2\arctan 2 = \frac{1}{2}\ln 5 - (-\arctan(-1) - \frac{1}{2}\ln 2 \\
&= 2\arctan 2 - \arctan 1 + \frac{1}{2}\ln\frac{2}{5}
\end{aligned}$$

Note: (*) and (**) are explained as following.
Let $x^2y^2 = \tan^2\theta \Rightarrow \theta = \arctan xy$. We know $1 + \tan^2\theta = \sec^2\theta$. And $\frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta d\theta$.
Thus,

$$\begin{aligned}
\int \frac{y}{x^2y^2}dx &= \int \frac{y}{\tan^2\theta + 1}\sec^2\theta d\theta \\
&= \int \frac{y}{\sec^2\theta}\sec^2\theta d\theta \\
&= \int y d\theta \\
&= \arctan xy
\end{aligned}$$

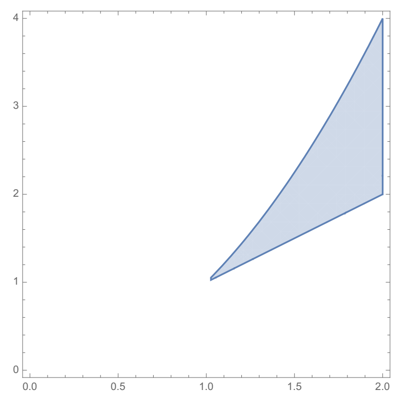
Let $u = \arctan y$ and $v' = 1$. Then using integration by parts,

$$\int \arctan y dy = y \arctan y - \int \frac{y}{1+y^2} dy$$
$$= y \arctan y - \frac{1}{2} \ln 1 + y^2$$

4. (5 points) Sketch the region of integration and evaluate the integral:

$$\int_1^4 \int_{\sqrt{y}}^{y} x^2 y^3 \; dx \; dy.$$

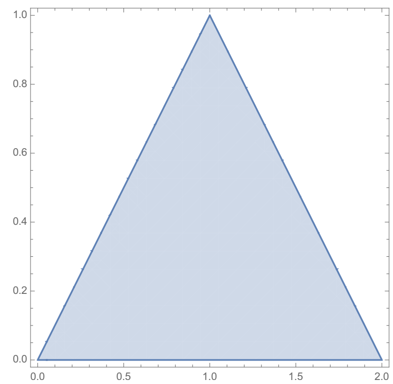
Solution: Skectch:



$$\begin{aligned} \int_1^4 \int_{\sqrt{y}}^{y} x^2 y^3 \; dx \; dy &= \int_1^4 \left. \frac{x^3 y^3}{3} \right|_{\sqrt{y}}^{y} dy \\ &= \frac{1}{3} \int_1^4 y^6 - y^{9/2} \; dy \\ &= \frac{1}{3} \left(\frac{y^7}{7} - \frac{y^{11/2}}{11/2} \right) \Bigg|_1^4 \\ &= \frac{4^7 - 1}{21} - \frac{4^{11/2} - 1}{33/2} = \frac{151555}{231} \approx 656.082. \end{aligned}$$

5. (5 points) (a) Sketch the region in the xy-plane that is bounded by the x-axis, $y = x$, and $x + y = 2$.

Solution: Skectch:



(b) Express the integral of $f(x, y)$ over this region in terms of iterated integrals in two ways. (That is, formulate the integral in two ways: in one, use $dx\ dy$; in the other, use $dy\ dx$.)

Solution: Let R denote the region described. Then,

$$\begin{aligned}\iint_R f(x,y)\ dA &= \int_0^1 \int_0^x f(x,y)\ dy\ dx + \int_1^2 \int_0^{2-x} f(x,y)\ dy\ dx \\ &= \int_0^1 \int_y^{2-y} f(x,y)\ dx\ dy\end{aligned}$$

(c) Using one of your answers to part (b), evaluate the integral exactly for $f(x, y) = x$.

Solution:

$$\begin{aligned}\iint_R x\ dA &= \int_0^1 \int_y^{2-y} x\ dx\ dy \\ &= \int_0^1 \left. \frac{x^2}{2} \right|_y^{2-y} dy \\ &= \frac{1}{2} \int_0^1 (4 - 2y + y^2) - y^2\ dy \\ &= \frac{1}{2} \left. \left(4y - y^2\right) \right|_0^1 = 3/2.\end{aligned}$$

6. (5 points) If R is the region $x + y \geq a$, $x^2 + y^2 \leq a^2$, with $a > 0$, evaluate the integral

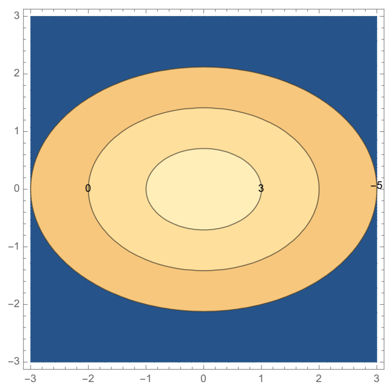
$$\int_R xy\ dA.$$

Solution: The two equations intersect at $(0, a)$ and $(a, 0)$. Thus,

$$\begin{aligned}\int_R xy\ dA &= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} xy\ dy\ dx \\ &= \frac{1}{2}\int_0^a (a^2x - x^3 - ax + x^2)\ dx \\ &= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4} - \frac{a^3}{2} + \frac{a^3}{3}\right) \\ &= \frac{a^3}{24}\end{aligned}$$

7. (5 points) (a) Sketch the level curves of the function $f(x, y) = 4 - x^2 - 2y^2$, at levels $k = 4, 3, 0$, and -5.

Solution: Sketch:



(b) What region R in the xy-plane maximizes the value of

$$\int\int_R (4 - x^2 - 2y^2)\ dA\ ?$$

Give reasons for your answer.

Solution: Interpret the double integral $\iint_R (4 - x^2 - 2y^2)\ dA$ as the volume "under the graph of $z = f(x, y)$", where $f(x, y) = 4 - x^2 - 2y^2$. If the graph $z = f(x, y)$ is below the xy-plane, then the volume will be counted as a negative number.

Note that $f(x, y)$ is nonnegative inside the ellipse $E = \{(x, y) \mid \frac{x^2}{2} + y^2 \leq 2\}$, and is negative outside it. So, the region in the xy-plane that maximizes the volume under the surface $z = f(x, y)$ is precisely the ellipse E above: choose

$$R = E = \{(x, y) \mid \frac{x^2}{2} + y^2 \leq 2\}.$$

(c) Then, express the double integral over the region R you specified above as an iterated integral.

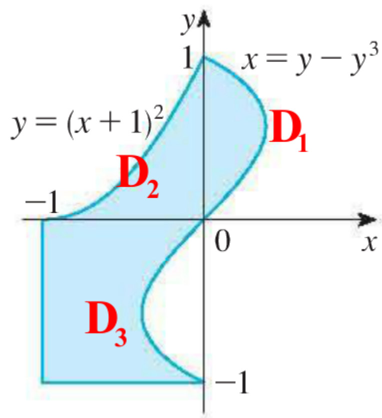
Solution: The range of the x-coordinates of points (x, y) in E is: $\frac{x^2}{2} \leq 2$, so $-2 \leq x \leq 2$. Then, for each x in this range, the points (x, y) is in E if y is in the range: $y^2 \leq 2 - \frac{x^2}{2}$, or

$$-\sqrt{2 - \frac{x^2}{2}} \leq y \leq \sqrt{2 - \frac{x^2}{2}}.$$

Hence, the double integral over R chosen above is:

$$\iint_R 4 - x^2 - 2y^2 \, dA = \int_{-2}^{2} \int_{-\sqrt{2-\frac{x^2}{2}}}^{\sqrt{2-\frac{x^2}{2}}} 4 - x^2 - 2y^2 \, dy \, dx.$$

8. (5 points) Express D, the shaded region below, as a union of regions of type I or type II and evaluate the integral $\int\int_D y \, dA$.



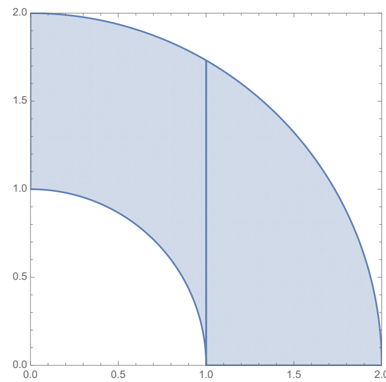
Solution:

$$\begin{aligned}
\iint_D y \, dA &= \iint_{D_1} y \, dA + \iint_{D_2} y \, dA + \iint_{D_3} y \, dA \\
&= \int_0^1 \int_0^{y-y^3} y \, dx \, dy + \int_{-1}^{0} \int_0^{(x+1)^2} y \, dy \, dx + \int_{-1}^{0} \int_{-1}^{y-y^3} y \, dx \, dy \\
&= \int_0^1 y(y - y^3) dy + \int_0^{-1} \frac{1}{2}(x+1)^4 dx + \int_{-1}^{0} (y(y-y^3) + y) dy \\
&= 2\int_{-1}^{1} y(y - y^3) dy + \int_0^{-1} \frac{1}{2}(x+1)^4 dx + \int_{-1}^{0} (y(y-y^3) + y) dy \\
&= 2\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) - \frac{1}{2} \\
&= -\frac{4}{30}
\end{aligned}$$

9. (5 points) (a) Sketch the region of integration of

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\, dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x\, dy\, dx.$$

Solution: Sketch:



(b) Evaluate the integral in part (a) by first converting into polar coordinates.

Solution:

$$\begin{aligned}
\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\, dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x\, dy\, dx &= \int_0^{\pi/2} \int_1^2 r\cos(\theta)\ r\ dr\ d\theta \\
&= \int_0^{\pi/2} \int_1^2 r^2 \cos(\theta)\ dr\ d\theta \\
&= \int_0^{\pi/2} \frac{r^3}{3}\cos(\theta)\Big|_1^2\ d\theta \\
&= \int_0^{\pi/2} \frac{7}{3}\cos(\theta)\ d\theta \\
&= \frac{7}{3}\sin(\theta)\Big|_0^{\pi/2} = \frac{7}{3}.
\end{aligned}$$

10. (5 points) Find the volume of an ice cream cone bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: First, find the intersection between the hemisphere and the ice cream cone:

$$8 - x^2 - y^2 = x^2 + y^2,\ z = \sqrt{x^2 + y^2}$$

or equivalently,

$$x^2 + y^2 = 4,\ z = 2$$

a circle of radius 2 parallel to the xy-plane, centered at $(0, 0, 2)$.

So, the region of integration, call it D, should be the circle of radius 2 in the xy-plane, centered at the origin. The equation in polar coordinate is: $r = 2$, $0 \leq \theta \leq 2$. The integrand should be the z-range so that for each (x, y), the point (x, y, z) is inside the ice cream cone:

$$\sqrt{x^2 + y^2} \leq z \leq \sqrt{8 - x^2 - y^2}.$$

Therefore, the volume is:

$$\begin{aligned}
\iint_D \sqrt{8-x^2-y^2} - \sqrt{x^2+y^2}\ dA &= \int_0^{2\pi}\int_0^2 (\sqrt{8-r^2}-r)\ r\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 \sqrt{8-r^2}\ r\ dr\ d\theta - \int_0^{2\pi}\int_0^2 r^2\ dr\ d\theta \\
&= 2\pi\frac{16\sqrt{2}}{3} - 2\pi\frac{8}{3} = \frac{16(2\sqrt{2}-1)\pi}{3}.
\end{aligned}$$