New York University
MATH.UA 123 Calculus 3

Problem Set 5

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (4 points) Sketch and describe the region of integration of the integral below. Include clear explanation/justification.

$$\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2-z^2}} f(x,y,z)\ dy\ dx\ dz.$$

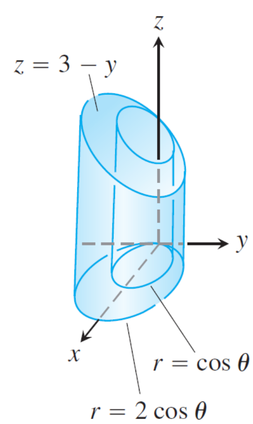
2. (4 points) Evaluate the following triple integral using only geometric interpretation and symmetry:

$$\iiint_C (4+5x^2yz^2)\ dV,$$

where C is the cylindrical region: $x^2+y^2 \le 4$, $-2 \le z \le 2$.

3. (4 points) Find the volume of the region bounded between the planes $z = 1+x+y$ and $x+y+z=1$, and above the triangle $x+y \le 1$, $x \ge 0$, $y \ge 0$ in the xy-plane.

4. (4 points) Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\ dz\ r\ dr\ d\theta$ over the region D, described as follows.

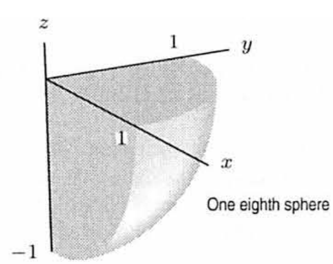


D is the solid cylinder whose base is the region in the xy-plane between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3-y$.

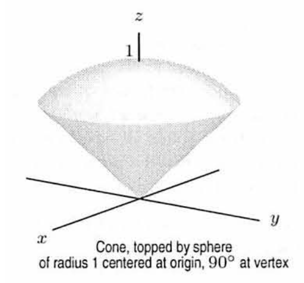
5. (4 points) A solid is bounded from below by the cone $z = \sqrt{x^2+y^2}$ and from above by the plane $z = 1$. The density of the solid is given by $\delta(r,\theta,z) = z^2$. Find the mass of and the average density of the solid.

6. (4 points) For each of the regions W shown below, write the limits of integration for $\int_W dV$ in the following coordinates: (1) Cartesian, (2) Cylindrical, and (3) Spherical.

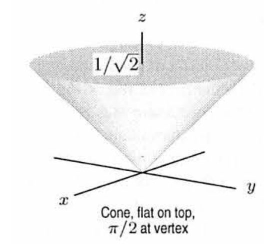
(a)



(b)

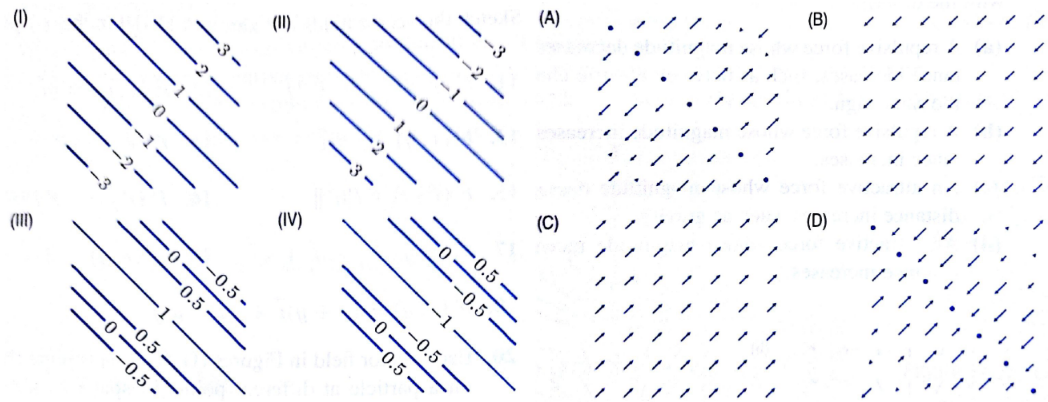


(c)

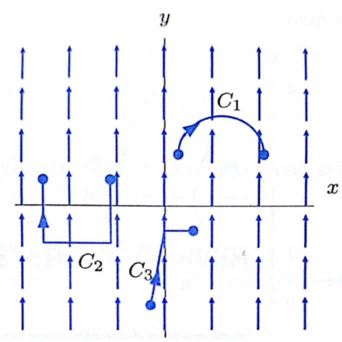


7. (3 points) Figures (I)-(IV) contain level curves of functions of of two variables $f(x, y)$. Figures (A)-(B) are their corresponding gradient fields $\nabla f(x, y)$.

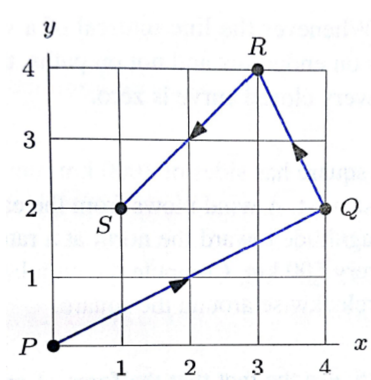
   Match the level curves in (I)-(IV) with the gradient fields in (A)-(D). All figures have $-2 \leq x \leq 2$, $-2 \leq y \leq 2$. Provide a brief explanation.



8. (3 points) Let $\mathbf{F}$ be the constant force field $\mathbf{j}$ in the figure to the right. On which of the paths C_1, C_2, C_3 is zero work done by $\mathbf{F}$?



9. (4 points) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the oriented curve in the figure to the right, and $\mathbf{F}$ is the vector field that is constant on each of the three straight segments of C:



$$\mathbf{F}(x,y) = \begin{cases} \mathbf{i} & \text{on } PQ \\ 2\mathbf{i} - \mathbf{j} & \text{on } QR \\ 3\mathbf{i} + \mathbf{j} & \text{on } RS. \end{cases}$$

10. (4 points) Along a curve C, a vector field $\mathbf{F}$ is everywhere tangent to C in the direction of orientation and has constant magintude $|\mathbf{F}| = m$.

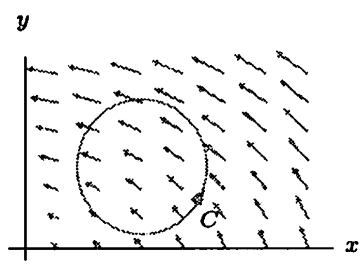
Use the definition of the line integral to explain why

$$\int_C \mathbf{F} \cdot d\mathbf{r} = m \cdot \text{ Length of } C.$$

11. (4 points) **Evaluating a work integral two ways**. Let $\mathbf{F} = \nabla(x^3y^2)$ and let C be the path in the xy-plane from $(-1,1)$ to $(1,1)$ that consists of the line segment from $(-1,1)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(1,1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.
    (a) Find parametrizations for the segments that make up C, and evaluate the integral.
    (b) Using $f(x,y) = x^3y^2$ as a potential function for $\mathbf{F}$.

12. (4 points) Show that the work done by a constant force field $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in moving a particle along any path from A to B is $W = \mathbf{F} \cdot \overrightarrow{AB}$.

13. (4 points) Consider the vector field $\mathbf{F}$ shown in the figure below.



    (a) Is $\oint_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero?
    (b) From your answer to part (A), can you determine whether or not $\mathbf{F} = \nabla f$ for some function f?
    (c) Which of the following formulas best fits $\mathbf{F}$?

$$\begin{aligned} \mathbf{F}_1 &= \frac{x}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j} \\ \mathbf{F}_2 &= -y\mathbf{i} + x\mathbf{j} \\ \mathbf{F}_3 &= \frac{-y}{(x^2+y^2)^2}\mathbf{i} + \frac{x}{(x^2+y^2)^2}\mathbf{j}. \end{aligned}$$