New York University
MATH.UA 123 Calculus 3

Problem Set 5

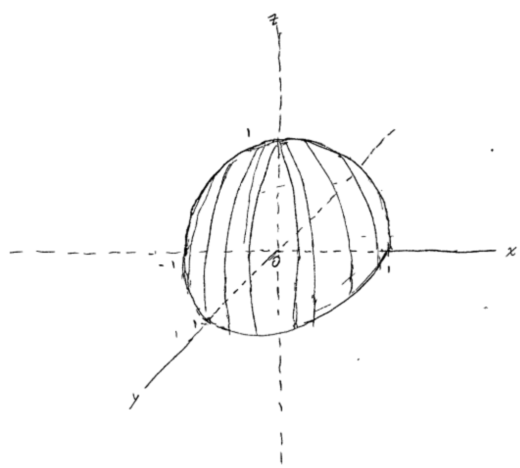
This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (4 points) Sketch and describe the region of integration of the integral below. Include clear explanation/justification.

$$\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2-z^2}} f(x,y,z)\ dy\ dx\ dz.$$

Solution: $0 \le z \le 1,\ -\sqrt{1-z^2} \le x \le \sqrt{1-z^2} \Leftrightarrow 0 \le x^2+z^2 \le 1,\ 0 \le y \le \sqrt{1-x^2-z^2} \Leftrightarrow 0 \le x^2+y^2+z^2 \le 1$. Combine the three inequalities, then



2. (4 points) Evaluate the following triple integral using only geometric interpretation and symmetry:

$$\iiint_C (4 + 5x^2yz^2)\ dV,$$

where C is the cylindrical region: $x^2+y^2 \le 4$, $-2 \le z \le 2$.

Solution: Split the integral into the sum of two integrals over C:

$$\iiint_C 4\ dV + 5\iiint_C x^2yz^2\ dV.$$

Note that $\iiint_C 4\ dV = 4\iiint_C 1\ dV = 4$ times the volume of C. Since C is a cylinder, its volume is the area of the base (circle of radius 2) with its height (4), namely $4 \times 2^2\pi \times 4 = 64\pi$.

Consider the second integral, $\iiint_C x^2yz^2\ dV$. Note that the cylinder C is symmetric about the xz-plane: if the point (x, y, z) is in the cylinder C, then $(x, -y, z)$ is also in C. On the other hand, the function $f(x, y, z) = x^2yz^2$ is odd with respect to y:

$$f(x, -y, z) = -f(x, y, z).$$

Let C_1 denote the part of C that has negative y components and C_2 the part of C that has negative y components. Then,

$$\begin{aligned} \iiint_C f(x,y,z)\ dV &= \iiint_{C_1} f(x,y,z)\ dV + \iiint_{C_2} f(x,y,z)\ dV \\ &= \iiint_{C_1} f(x,y,z)\ dV - \iiint_{C_1} f(x,y,z)\ dV = 0. \end{aligned}$$

So, $\iiint_C (4 + 5x^2yz^2)\ dV = \iiint_C 4\ dV = 64\pi$.

3. (4 points) Find the volume of the region bounded between the planes $z = 1 + x + y$ and $x + y + z = 1$, and above the triangle $x + y \leq 1$, $x \geq 0$, $y \geq 0$ in the xy-plane.

Solution:

The easiest way to think about this, is to think of the planes $z = 1 + x + y$ and $x + y + z = 1$ as the graph of two functions of (x, y). Then, the region in the xy-plane that we are integrating over is:

$$R = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq 1\}$$

which can be written as:

$$R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\} \tag{1}$$

or

$$R = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq 1 - y\}. \tag{2}$$

Consider describing R as given in equation (1): For each (x, y) in R, we want to account for the region between the two planes. That is, below the plane $z = 1 + x + y$ and above the plane $z = 1 - x - y$. So, the corresponding three dimensional region is:

$$\{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 1 - x - y \leq z \leq 1 + x + y\}.$$

Therefore, the corresponding integral is:

$$\int_0^1 \int_0^{1-x} \int_{1-x-y}^{1+x+y} f(x, y, z)\ dz\ dy\ dx$$

By describing R as given in equation (2) and using similar reasoning, we can write a second iterated integral:

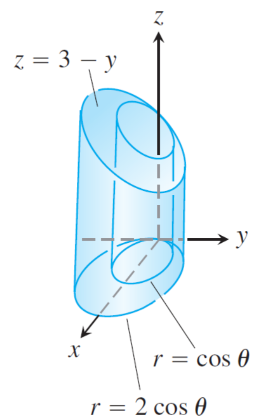
$$= \int_0^1 \int_0^{1-y} \int_{1-x-y}^{1+x+y} f(x, y, z)\ dz\ dx\ dy$$

Additionally, here are the other four other (less obvious) iterated integrals:

$$= \int_0^1 \int_0^1 \int_{1-z-y}^{1-y} f(x, y, z)\ dx\ dz\ dy + \int_0^1 \int_1^2 \int_{z-1-y}^{1-y} f(x, y, z)\ dx\ dz\ dy$$

$$= \int_0^1 \int_0^1 \int_{1-z-y}^{1-y} f(x,y,z)\ dx\ dy\ dz + \int_1^2 \int_0^1 \int_{z-1-y}^{1-y} f(x,y,z)\ dx\ dy\ dz$$
$$= \int_0^1 \int_0^1 \int_{1-z-x}^{1-x} f(x,y,z)\ dy\ dz\ dx + \int_0^1 \int_1^2 \int_{z-1-x}^{1-x} f(x,y,z)\ dy\ dz\ dx$$
$$= \int_0^1 \int_0^1 \int_{1-z-x}^{1-x} f(x,y,z)\ dy\ dx\ dz + \int_1^2 \int_0^1 \int_{z-1-x}^{1-x} f(x,y,z)\ dy\ dx\ dz$$

4. (4 points) Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\ dz\ r\ dr\ d\theta$ over the region D, described as follows.



D is the solid cylinder whose base is the region in the xy-plane between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$.

Solution:

$$\int_{-\pi/2}^{\pi/2} \int_{\cos\theta}^{2\cos\theta} \int_0^{3-r\sin\theta} f(r,\theta,z) r\ dz\ dr\ d\theta.$$

5. (4 points) A solid is bounded from below by the cone $z = \sqrt{x^2+y^2}$ and from above by the plane $z = 1$. The density of the solid is given by $\delta(r,\theta,z) = z^2$. Find the mass of and the average density of the solid.

Solution: Since $\sqrt{x^2+y^2} \le z \le 1$,

$$\begin{aligned} \text{Mass} &= \iiint z^2 dV \\ &= \int_0^{2\pi} \int_0^1 \int_r^1 z^2 r\ dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(\frac{r}{3} - \frac{r^4}{3}\right) dr d\theta \\ &= \int_0^{2\pi} \frac{1}{10}\ d\theta \\ &= \frac{\pi}{5} \end{aligned}$$

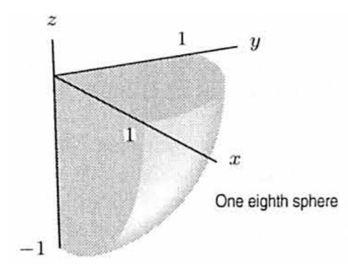
Compute the volume,

$$\begin{aligned}
\text{Volume} &= \iiint dV \\
&= \int_0^{2\pi} \int_0^1 \int_r^1 r \; dz dr d\theta \\
&= \int_0^{2\pi} \int_0^1 (r - r^2) \; dr d\theta \\
&= \int_0^{2\pi} \frac{1}{6} \; d\theta \\
&= \frac{\pi}{3}
\end{aligned}$$

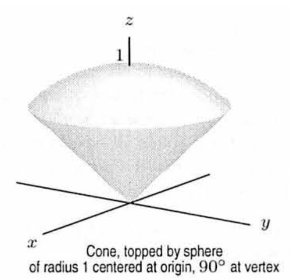
Thus, the average density of the solid is $(\pi/5)/(\pi/3) = 3/5$.

6. (4 points) For each of the regions W shown below, write the limits of integration for $\int_W dV$ in the following coordinates: (1) Cartesian, (2) Cylindrical, and (3) Spherical.

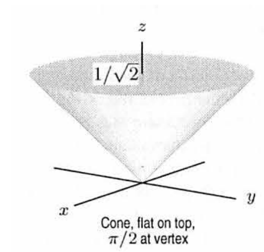
(a)



(b)



(c)



Solution: For (a),
The Cartesian is

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^0 \; dzdydz$$

We know $0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le 1$, $-\sqrt{1-x^2-y^2} \le z \le 0 \Leftrightarrow -\sqrt{1-r^2} \le z \le 0$. Thus,

$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_{-\sqrt{1-r^2}}^0 r \; dzdrd\theta$$

We know $\frac{\pi}{2} \le \phi \le \pi$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \rho \le 1$. Thus,

$$\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^1 \rho^2 \sin\phi \; d\rho d\phi d\theta$$

For (b),
The solid is lower bounded by $z = \sqrt{x^2+y^2}$ and upper bounded by $x^2+y^2+z^2 = 1$. Thus, $\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}$. Since $z = \sqrt{x^2+y^2}$, plug this in $x^2+y^2+z^2 = 1$. Then

$2x^2 + 2y^2 = 1 \Leftrightarrow x^2 + y^2 = 1/2 \Leftrightarrow y = \pm\sqrt{1/2 - x^2}$. Thus, $-\sqrt{1/2 - x^2} \le y \le \sqrt{1/2 - x^2}$. And from the graph, we know $-1 \le x \le 1$. Thus, the Cartesian is

$$\int_{-1}^{1} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} \; dzdydx$$

Since $\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}, r \le z \le \sqrt{1-r^2}$. From the graph, we know $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Thus,

$$\int_{0}^{2\pi} \int_{0}^{1} \int_{r}^{\sqrt{1-r^2}} r \; dzdrd\theta$$

Since $z = \sqrt{x^2+y^2}$, $\rho\cos\phi = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r = \rho\sin\phi$. Thus, $\phi = \pi/4$. From the graph, we know $0 \le \rho \le 1$ and $0 \le \theta \le 2\pi$, Thus,

$$\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{4}} \int_{0}^{1} \rho^2 \sin\phi \; d\rho d\phi d\theta$$

For (c),
The solid is lower bounded by $\sqrt{x^2+y^2}$ and upper bounded by $1/\sqrt{2}$. Thus, $\sqrt{x^2+y^2} \le z \le 1/\sqrt{2}$ Since $z = 1/\sqrt{2}$, plug this in $z = \sqrt{x^2+y^2}$. Then $x^2 + y^2 = 1/2 \Leftrightarrow y = \pm\sqrt{1/2 - x^2}$. Thus, $-\sqrt{1/2 - x^2} \le y \le \sqrt{1/2 - x^2}$. Also from the graph, we know $-1/\sqrt{2} \le x \le 1/\sqrt{2}$. Thus,

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\frac{1}{\sqrt{2}}} \; dzdydx$$

Since $0 \le r \le 1/\sqrt{2},\; 0 \le \theta \le 2\pi,\; r \le z \le 1/\sqrt{2}$,

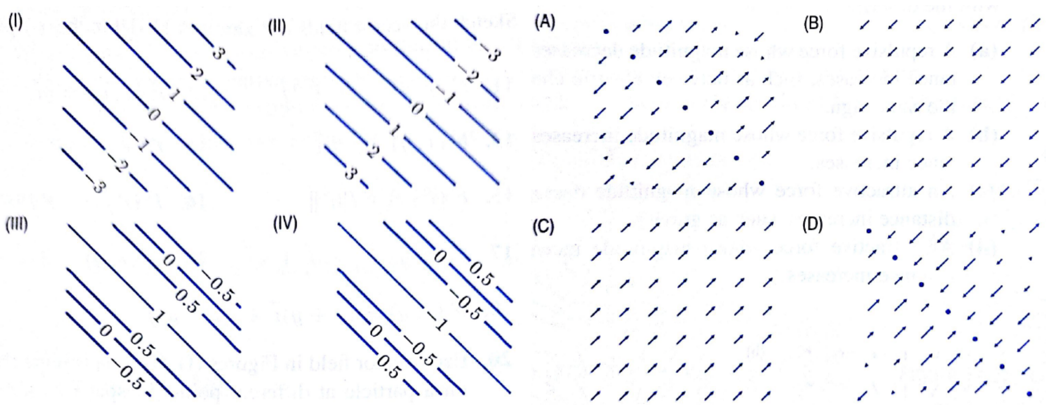
$$\int_{0}^{2\pi} \int_{0}^{\frac{1}{\sqrt{2}}} \int_{r}^{\frac{1}{\sqrt{2}}} r \; dzdrd\theta$$

From the graph, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/4$. The flat top is $1/2 = x^2 + y^2$. Rewrite this then, $1/2 = r^2 \Leftrightarrow 1/2 = \rho^2\cos^2\phi \Leftrightarrow 1/(\sqrt{2}\cos\phi) = \rho$. Thus,

$$\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{4}} \int_{0}^{\frac{1}{\sqrt{2}\cos\phi}} \rho^2 \sin\phi \; d\rho d\phi d\theta$$

7. (3 points) Figures (I)-(IV) contain level curves of functions of of two variables $f(x, y)$. Figures (A)-(B) are their corresponding gradient fields $\nabla f(x, y)$.

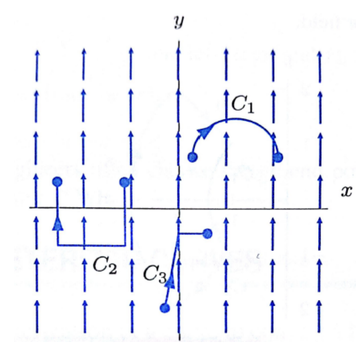
   Match the level curves in (I)-(IV) with the gradient fields in (A)-(D). All figures have $-2 \le x \le 2$, $-2 \le y \le 2$. Provide a brief explanation.



Solution: Explanation: The gradient of f at a point (x_0, y_0) where $f(x_0, y_0) = k$ is orthogonal to the level curve $f(x, y) = k$ and points in the direction of increasing f. Therefore:

- I - C
- II - B
- III - D
- IV - A

8. (3 points) Let $\mathbf{F}$ be the constant force field $\mathbf{j}$ in the figure to the right. On which of the paths C_1, C_2, C_3 is zero work done by $\mathbf{F}$?

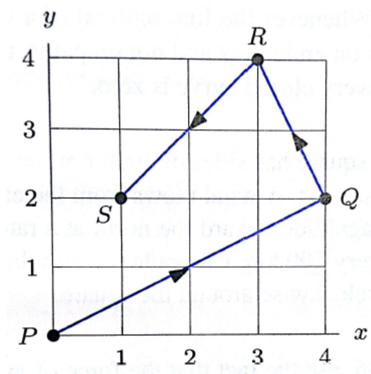


Solution: Total work is zero on C_1, C_2.

First consider the path C_2, which is piecewise smooth. We claim that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$. On the first piece, pointing downwards, work is negative since the constant force field $\mathbf{F}$ is antiparallel to the direction of movement of the particle. Then, on the second piece, work is zero since the particle is moving in a direction that is perpendicular to the force. Finally, on the third piece, the work is positive. Since the first and third piece are antiparallel and of the same length, and since the vector field is constant, then the work done on the first and third pieces cancel out one another, resulting in a total work of zero along C_2.

The path C_1 is the upper half of a circle. Splitting C_1 into the left half and the right half, we can use similar arguments as above to see that the work done on the left half of C_1 cancels out the work done on the right half of C_1, resulting in a total work of zero along C_1.

9. (4 points) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the oriented curve in the figure to the right, and $\mathbf{F}$ is the vector field that is constant on each of the three straight segments of C:



$$\mathbf{F}(x,y) = \begin{cases} \mathbf{i} & \text{on } PQ \\ 2\mathbf{i} - \mathbf{j} & \text{on } QR \\ 3\mathbf{i} + \mathbf{j} & \text{on } RS. \end{cases}$$

Solution: First, compute $\int_{PQ} \mathbf{F} \cdot d\mathbf{r}$. Along PQ, the curve is parametrized by: $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j}$, for $t = 0$ to $t = 2$. So,

$$\begin{aligned} \int_{PQ} \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 (\mathbf{i} + 0\mathbf{j}) \cdot (2\mathbf{i} + \mathbf{j})\, dt \\ &= \int_0^2 2\, dt = 4. \end{aligned}$$

Next, compute $\int_{QR} \mathbf{F} \cdot d\mathbf{r}$. Along QR, the curve is parametrized by: $\mathbf{r}(t) = (4-t)\mathbf{i} + (2+2t)\mathbf{j}$, for $t = 0$ to $t = 1$. So,

$$\begin{aligned} \int_{QR} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (2\mathbf{i} - \mathbf{j}) \cdot (-\mathbf{i} + 2\mathbf{j})\, dt \\ &= \int_0^1 -4\, dt = -4. \end{aligned}$$

Finally, compute $\int_{RS} \mathbf{F} \cdot d\mathbf{r}$. Along RS, the curve is parametrized by: $\mathbf{r}(t) = (3-2t)\mathbf{i} + (4-2t)\mathbf{j}$, for $t = 0$ to $t = 1$. So,

$$\begin{aligned} \int_{RS} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (3\mathbf{i} + \mathbf{j}) \cdot (-2\mathbf{i} - 2\mathbf{j})\, dt \\ &= \int_0^1 -8\, dt = -8. \end{aligned}$$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 4 - 4 - 8 = -8$.

10. (4 points) Along a curve C, a vector field $\mathbf{F}$ is everywhere tangent to C in the direction of orientation and has constant magintude $|\mathbf{F}| = m$.

Use the definition of the line integral to explain why

$$\int_C \mathbf{F} \cdot d\mathbf{r} = m \cdot \text{ Length of } C.$$

Solution: Suppose that C is parametrized by $\mathbf{r}(t)$ for $t = a$ to $t = b$. Since $\mathbf{F}$ is everywhere tangent to C in the direction of orientation, then $\mathbf{F}$ and $\mathbf{r}'$ points in the same direction. Therefore, the unit vector in direction $\mathbf{F}$ and the unit vector in direction $\mathbf{r}'$ must in fact be equal,

$$\frac{\mathbf{F}}{|\mathbf{F}|} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|},$$

for all t between a and b. So,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\ |\mathbf{r}'(t)|\ dt \\
&= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{F}(t)}{|\mathbf{F}(t)|}\ |\mathbf{r}'(t)|\ dt \\
&= \int_a^b \frac{|\mathbf{F}(\mathbf{r}(t))|^2}{|\mathbf{F}(\mathbf{r}(t))|}\ |\mathbf{r}'(t)|\ dt \\
&= \int_a^b m\ |\mathbf{r}'(t)|\ dt = m \int_a^b |\mathbf{r}'(t)|\ dt,
\end{aligned}$$

which is m times the arclength of C.

11. (4 points) **Evaluating a work integral two ways**. Let $\mathbf{F} = \nabla(x^3y^2)$ and let C be the path in the xy-plane from $(-1,1)$ to $(1,1)$ that consists of the line segment from $(-1,1)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(1,1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.

(a) Find parametrizations for the segments that make up C, and evaluate the integral.

Solution: Let C_1 denote the line segment from $(-1,1)$ to $(0,0)$ and C_2 the line segment from $(0,0)$ to $(1,1)$.
One possible parametrization of C_1:

$$\mathbf{r}(t) = (-1+t)\mathbf{i} + (1-t)\mathbf{j}, \quad 0 \le t \le 1.$$

One possible parametrization of C_2:

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1.$$

The vector field is $\mathbf{F} = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$. So,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^1 (3(-1+t)^2(1-t)^2\mathbf{i} + 2(-1+t)^3(1-t)\mathbf{j}) \cdot (1\mathbf{i} - 1\mathbf{j})\ dt \\
&\quad + \int_0^1 (3t^2t^2\mathbf{i} + 2t^3t\mathbf{j}) \cdot (1\mathbf{i} + 1\mathbf{j})\ dt \\
&= \int_0^1 5(1-t)^4\ dt + \int_0^1 5t^4\ dt \\
&= -(1-t)^5\Big|_0^1 + t^5\Big|_0^1 = 2.
\end{aligned}$$

(b) Using $f(x,y) = x^3y^2$ as a potential function for $\mathbf{F}$.

Solution: Since $\mathbf{F} = \nabla f$, by the fundamental theorem of line integrals,

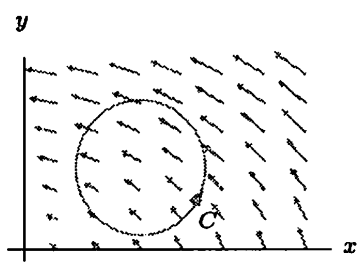
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= f(1,1) - f(-1,1) \\ &= 1-(-1) = 2.\end{aligned}$$

12. (4 points) Show that the work done by a constant force field $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in moving a particle along any path from A to B is $W = \mathbf{F}\cdot\overrightarrow{AB}$.

Solution:

$$\begin{aligned}\int_A^B \mathbf{F}\cdot d\mathbf{r} &= \int_{t_1}^{t_2} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt \\ &= \int_{t_1}^{t_2} ax'(t) + by'(t) + cz'(t)dt \\ &= a\int_{t_1}^{t_2} x'(t)dt + b\int_{t_1}^{t_2} y'(t)dt + c\int_{t_1}^{t_2} z'(t)dt \\ &= a(x(t_2)-x(t_1)) + b(y(t_2)-y(t_1)) + c(z(t_2)-z(t_1)) \\ &= \mathbf{F}\cdot\overrightarrow{AB}\end{aligned}$$

13. (4 points) Consider the vector field $\mathbf{F}$ shown in the figure below.



(a) Is $\oint_C \mathbf{F}\cdot d\mathbf{r}$ positive, negative, or zero?

Solution: Positive

(b) From your answer to part (A), can you determine whether or not $\mathbf{F} = \nabla f$ for some function f?

Solution: If $\mathbf{F} = \nabla f$ for some function f, then $\mathbf{F}$ is conservative. By the fundamental theorem of line integrals, $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ over any closed curve C. Since in part (a) we showed that this line integral is strictly positive, then $\mathbf{F}$ is not conservative.

(c) Which of the following formulas best fits $\mathbf{F}$?

$$\begin{aligned}\mathbf{F}_1 &= \frac{x}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j} \\ \mathbf{F}_2 &= -y\mathbf{i} + x\mathbf{j} \\ \mathbf{F}_3 &= \frac{-y}{(x^2+y^2)^2}\mathbf{i} + \frac{x}{(x^2+y^2)^2}\mathbf{j}.\end{aligned}$$

Solution: $\mathbf{F}_3$ best fits $\mathbf{F}$, since the magnitude of the vectors are longer the farther away they are from the origin (thus, $\mathbf{F}_1$ or $\mathbf{F}_3$), and since the directions of the vectors are consistent with the signs of $\mathbf{F}_2$ and $\mathbf{F}_3$, namely horizontally to the left and vertically upward, for all points in the positive orthant.