New York University
MATH.UA 123 Calculus 3

Problem Set 6

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (4 points) Show that the value of

$$\oint_C xy^2 \; dx + (x^2y + 2x) \; dy$$

around any square depends only on the area of the square and not on its location in the plane.

2. (4 points) (a) For which of the following can you use Green's Theorem to evaluate the integral? Explain.

   I. $\int_C (x^2+y^2)dx + (x^2+y^2)dy$ where C is the boundary of the region bounded by $y = x$, $y = x^2$, $0 \le x \le 1$, with counterclockwise orientation.

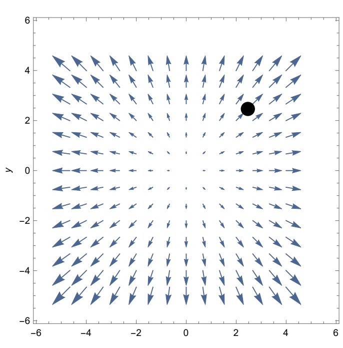
   II. $\int_C \frac{1}{\sqrt{x^2+y^2}}dx - \frac{1}{\sqrt{x^2+y^2}}dy$ where C is the unit circle centered at the origin, oriented counterclockwise.

   III. $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and C is the line segment from the origin to the point $(1, 1)$.

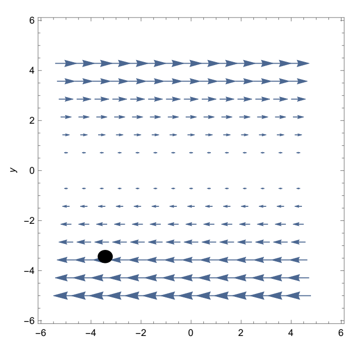
   (b) Use Green's Theorem to evaluate the integrals in part (a) that can be done that way.

3. (4 points) For each of the following vector fields, determine if the divergence is positive, zero, or negative at the indicated point. Explain/justify your answer.

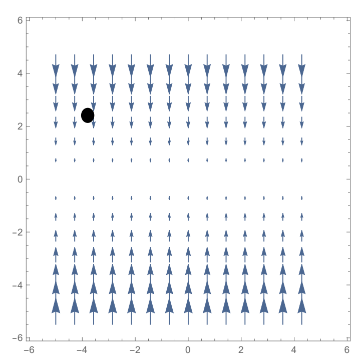
(a)



(b)



(c)



4. (4 points) (a) Sketch a the vector field $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ in the xy-plane.

   (b) Based on your sketch, what is the direction of rotation of a thin twig placed at the origin along the x-axis?

   (c) Based on your sketch, what is the direction of rotation of a thin twig placed at the origin along the y-axis?

   (d) Compute curl$\mathbf{F}$.

5. (4 points) Prove each identity below, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then $\mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are defined by

$$\begin{aligned}(\mathbf{F} \cdot \mathbf{G})(x, y, z) &= \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z) \\ (\mathbf{F} \times \mathbf{G})(x, y, z) &= \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)\end{aligned}$$

(a) $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl}(\mathbf{F}) - \mathbf{F} \cdot \operatorname{curl}(\mathbf{G})$

(b) $\operatorname{curl}(\operatorname{curl}(\mathbf{F})) = \operatorname{grad}(\operatorname{div}(\mathbf{F})) - \nabla^2 \mathbf{F}$

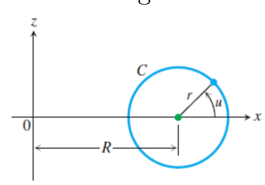
6. (5 points) Find a parametrization of the portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$. Use the parametrization to formulate the area of the surface as a double integral. Then, evaluate the integral.

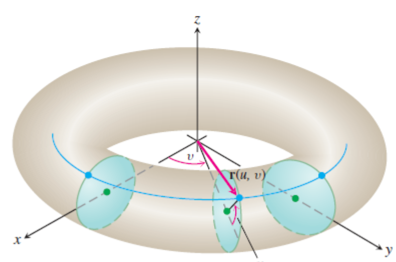
7. (5 points) Find a parametrization of the portion of the cone $z = \sqrt{x^2 + y^2}/3$ between the planes $z = 1$ and $z = 4/3$. Use the parametrization to formulate the area of the surface as a double integral. Then, evaluate the integral.

8. (5 points) *A torus of revolution* (doughnut) is obtained by rotating a circle C in the xz plane about the z-axis. Suppose that C has a radius r and center $(R, 0, 0)$.

(a) Find a parametrization $\mathbf{r}(u, v)$ of the torus. Specify the set D in which (u, v) must lie.

Hint: You can choose let u represent the angle that the line from the point $\mathbf{r}(u, v)$ on the torus to the center of the rotated circle form with the xy-plane, and let v denote the angle formed by the line from the point $\mathbf{r}(u, v)$ on the torus to the origin with the positive x-axis. See figures below.





(b) Show that the surface area of the torus is $4\pi^2 Rr$.

9. (5 points) Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the surface S that is the portion of the surface $y^2 + 4z = 16$ that lies between the planes $x = 0$, $x = 1$, and $z = 0$.

10. (5 points) Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$, where S is the helicoid $\mathbf{r}(u, v) = \langle u\cos(v), u\sin(v), v\rangle$, $0 \leq u \leq 1, 0 \leq v \leq \pi$, with upward orientation.

11. (5 points) Find the outward flux of the field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the surface of the portion of the sphere $x^2 + y^2 + z^2 \leq 25$ above the plane $z = 3$.

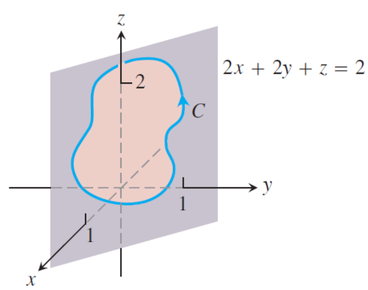
The following problems are important, but they will not be graded. It is encouraged you work through them.

12. Use the surface integral in Stokes' Theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$ and C is thee intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$, counterclockwise when viewed from above.

13. Let $\mathbf{n}$ be the outer unit normal of the surface S given by $4x^2 + 9y^2 + 36z^2 = 26, z \geq 0$, and let $\mathbf{F}(x, y, z) = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2}\sin(e^{\sqrt{xyz}})\mathbf{k}$.

    Find the value of
    $$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}.$$
    Hint: One parametrization of the ellipse at the base of the shell is of the form $x = a\cos(t), y = b\cos(t)$, for some constants a, b.

14. **Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented as shown here. Show that
    $$\int_C 2y\ dx + 3z\ dy - x\ dz$$
    depends only on the area of the region enclosed by C and not on the position or shape of C.



15. Use the divergence theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D, where $\mathbf{F}(x, y, z) = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$ and D is the region inside the solid cylinder $x^2 + y^2 \leq 4$, between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$.

16. **Among all rectangular solids defined by the inequalities $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq 1$, find the one for which the total flux of $\mathbf{F}(x, y, z) = (-x^2 - 4xy)\mathbf{i} - 6yz\mathbf{j} + 12z\mathbf{k}$ outward through the six sides is the greatest. What is the value of the greatest flux?