New York University
MATH.UA 123 Calculus 3

Problem Set 6

This problem set consists not only of problems similar to what you've seen, but also of unique problems you may not have seen before. The purpose of the latter is for you to apply the concepts you've previously learned to new, unfamiliar, and usually more interesting situations. In some cases, problems connect ideas from multiple learning objectives.

Write full, clear solutions to the problems below. It is important that the logic of how you solved these problems is clear. Although the final answer is important, being able to convey you understand the underlying concepts is more important. The point weight of each problem is indicated prior to each question. This problem set is graded out of 50 total points.

1. (4 points) Show that the value of

$$\oint_C xy^2 \ dx + (x^2y + 2x) \ dy$$

around any square depends only on the area of the square and not on its location in the plane.

Solution: Let D denote the square area bounded by the sides C. Then using Green's Theorem,

$$\begin{aligned}
\oint_C xy^2 \ dx + (x^2y + 2x) \ dy &= \iint_D \frac{\partial}{\partial x}(x^2y + 2x) + \frac{\partial}{\partial y}(-xy^2) \ dA \\
&= \iint_D (2xy + 2 - 2xy) \ dA \\
&= \iint_D 2 \ dA = 2\iint_D 1 \ dA \\
&= 2 \text{ times the area of } D.
\end{aligned}$$

2. (4 points) (a) For which of the following can you use Green's Theorem to evaluate the integral? Explain.

I. $\int_C (x^2 + y^2)dx + (x^2 + y^2)dy$ where C is the boundary of the region bounded by$y = x$, $y = x^2$, $0 \leq x \leq 1$, with counterclockwise orientation.

II. $\int_C \frac{1}{\sqrt{x^2+y^2}}dx - \frac{1}{\sqrt{x^2+y^2}}dy$ where C is the unit circle centered at the origin, oriented counterclockwise.

III. $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and C is the line segment from the origin to the point $(1, 1)$.

Solution: We can use Green's Theorem for I since D is a simple closed curve that is piecewise smooth (and positively-oriented), and P and Q have continuous partial derivatives (everywhere on $\mathbb{R}^2$).

We cannot use Green's Theorem for II since P and Q (and their partial derivatives) are not continuous at $(0, 0)$, which is inside the region bounded by C.

We cannot use Green's Theorem for III directly since C is not a closed curve.

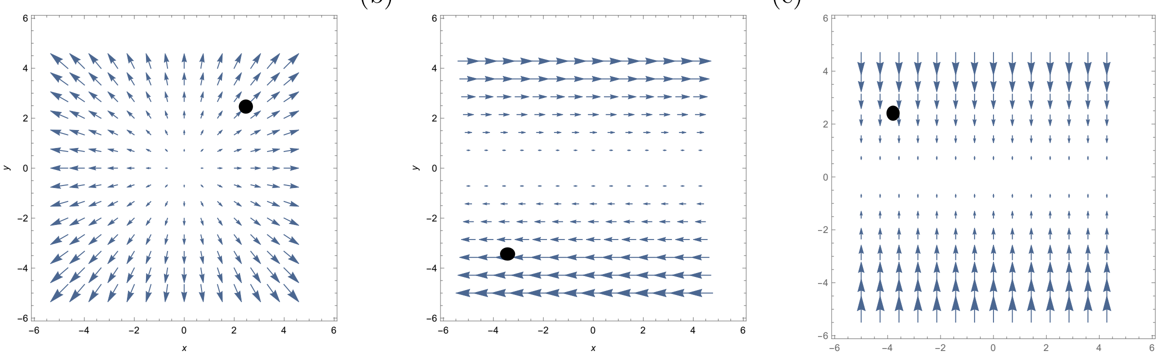
(b) Use Green's Theorem to evaluate the integrals in part (a) that can be done that way.

Solution: For I, let D denote the region bounded by C. Then, using Green's Theorem,

$$\begin{aligned}
\int_C (x^2+y^2)\;dx + (x^2+y^2)\;dy &= \iint_D 2x - 2y\;dA \\
&= \int_0^1 \int_{x^2}^{x} 2x - 2y\;dy\;dx \\
&= \int_0^1 2xy\big|_{x^2}^{x} - y^2\big|_{x^2}^{x}\;dx \\
&= \int_0^1 2x^2 - 2x^3 - x^2 + x^4\;dx \\
&= \frac{1}{30}
\end{aligned}$$

3. (4 points) For each of the following vector fields, determine if the divergence is positive, zero, or negative at the indicated point. Explain/justify your answer.

(a) (b) (c)

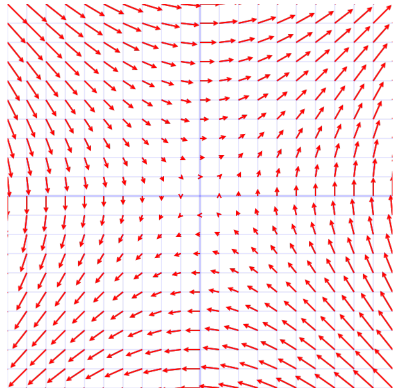


Solution:

(a) At the point, the arrows are pointing positive for both x and y. Thus, div $\mathbf{F} = ++$. Thus, it is positive. (b) At the point, the arrow is pointing negative for x. Thus, div $\mathbf{F} = -0$ so it is negative. (c) At the point, the arrow is pointing negative for y. Thus, div$\mathbf{F} = 0-$ so it is negative.

4. (4 points) (a) Sketch a the vector field $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ in the xy-plane.

Solution:



(b) Based on your sketch, what is the direction of rotation of a thin twig placed at the origin along the x-axis?

> **Solution:**
> Along the x-axis, for $x \in (0, \infty)$, the arrows are pointing upward and for $x \in (-\infty, 0)$, the arrows are pointing downward. Thus, the twig would rotate counterclockwise.

(c) Based on your sketch, what is the direction of rotation of a thin twig placed at the origin along the y-axis?

> **Solution:**
> Along the y-axis, for $y \in (0, \infty)$, the arrows are pointing right and $y \in (-\infty, 0)$, the arrows are pointing left. Thus, the twig would rotate clockwise.

(d) Compute curl$\mathbf{F}$.

> **Solution:**
>
> $$\text{curl } \mathbf{F} = (0 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (1 - 1)\mathbf{k} = \mathbf{0}$$

5. (4 points) Prove each identity below, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then $\mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are defined by

$$\begin{aligned} (\mathbf{F} \cdot \mathbf{G})(x, y, z) &= \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z) \\ (\mathbf{F} \times \mathbf{G})(x, y, z) &= \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z) \end{aligned}$$

(a) $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}(\mathbf{F}) - \mathbf{F} \cdot \text{curl}(\mathbf{G})$

> **Solution:** Let $\mathbf{F} = \langle P, Q, R \rangle$ and $G = \langle S, T, U \rangle$. Then
>
> $$\begin{aligned} \text{div}(\mathbf{F} \times \mathbf{G}) &= \text{div}(\langle QU - TR, RS - PU, PT - QS \rangle) \\ &= \frac{\partial}{\partial x}(QU - TR) + \frac{\partial}{\partial y}(RS - PU) + \frac{\partial}{\partial z}(PT - QS) \end{aligned}$$

$$\begin{aligned}
\mathbf{G}\cdot \mathrm{curl}(\mathbf{F}) &= \mathbf{G}\cdot(\nabla\times F)\\
&= \langle S,T,U\rangle\cdot\langle \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\rangle\\
&= S\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+T\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+U\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\rangle\right)
\end{aligned}$$

Similarly,

$$\mathbf{F}\cdot \mathrm{curl}(\mathbf{G}) = P\left(\frac{\partial U}{\partial y}-\frac{\partial T}{\partial z}\right)+Q\left(\frac{\partial S}{\partial z}-\frac{\partial U}{\partial x}\right)+R\left(\frac{\partial T}{\partial x}-\frac{\partial S}{\partial y}\right)$$

Then

$$\begin{aligned}
\mathbf{G}\cdot \mathrm{curl}(\mathbf{F})-\mathbf{F}\cdot \mathrm{curl}(\mathbf{G}) &= U\frac{\partial Q}{\partial x}+Q\frac{\partial U}{\partial x}-R\frac{\partial T}{\partial x}-T\frac{\partial R}{\partial x}\\
&\quad +S\frac{\partial R}{\partial y}+R\frac{\partial S}{\partial y}-U\frac{\partial P}{\partial y}-P\frac{\partial U}{\partial y}\\
&\quad +T\frac{\partial P}{\partial z}+P\frac{\partial T}{\partial z}-S\frac{\partial Q}{\partial z}-Q\frac{\partial S}{\partial z}\\
&= \frac{\partial}{\partial x}(QU-TR)+\frac{\partial}{\partial y}(RS-PU)+\frac{\partial}{\partial z}(PT-QS)\\
&= \mathrm{div}(\mathbf{F}\times\mathbf{G})
\end{aligned}$$

(b) $\mathrm{curl}(\mathrm{curl}(\mathbf{F})) = \mathrm{grad}(\mathrm{div}(\mathbf{F}))-\nabla^2\mathbf{F}$

Solution:

$$\begin{aligned}
\mathrm{curl}(\mathrm{curl}(\mathbf{F})) &= \mathrm{curl}(\nabla\times\mathbf{F})\\
&= \nabla\times(\nabla\times\mathbf{F})\\
&= \nabla(\nabla\cdot\mathbf{F})-(\nabla\cdot\nabla)(\mathbf{F})\\
&= \mathrm{grad}(\mathrm{div}(\mathbf{F}))-\nabla^2\mathbf{F}
\end{aligned}$$

6. (5 points) Find a parametrization of the portion of the plane $y+2z=2$ inside the cylinder $x^2+y^2=1$. Use the parametrization to formulate the area of the surface as a double integral. Then, evaluate the integral.

Solution: In cylindrical coordinates,

$$(x,y,z)\mapsto(\cos r\theta, \sin r\theta, z)$$

Then we have

$$x^2+y^2+z^2\le 1 \Longrightarrow r\le 1$$

and

$$z = 1-\frac{1}{2}y = 1-\frac{r}{2}\sin\theta$$

Hence, the surface can be parametrized as

$$S(\theta,r) = (r\cos\theta, r\sin\theta, 1-\frac{r}{2}\sin\theta)$$

where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ Then the arc is

$$A = \iint dS = \int_0^1 \int_0^{2\pi} \|s_\theta \times s_r\| d\theta dr$$
$$= \int_0^1 \int_0^{2\pi} \frac{\sqrt{5}}{2} r d\theta dr$$
$$= \frac{\sqrt{5}}{2}\pi$$

7. (5 points) Find a parametrization of the portion of the cone $z = \sqrt{x^2+y^2}/3$ between the planes $z = 1$ and $z = 4/3$. Use the parametrization to formulate the area of the surface as a double integral. Then, evaluate the integral.

Solution: Using the cylindrical coordinate,

$$z = \frac{\sqrt{x^2+y^2}}{3} \to z = \frac{r}{3}$$

So

$$(x, y, z) \mapsto (3z\cos\theta, 3z\sin\theta, z)$$

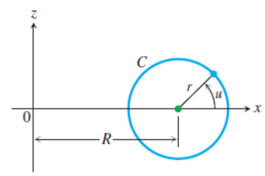
where $0 \leq \theta \leq 2\pi$ and $1 \leq z \leq \frac{4}{3}$. Call this $S(\theta, z)$. The surface area is then

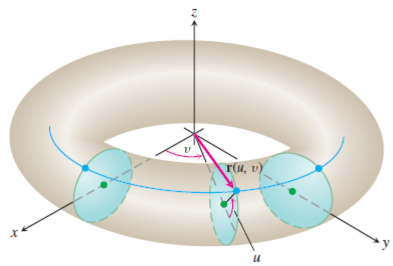
$$A = \int_0^{2\pi} \int_0^{\frac{4}{3}} \|s_\theta \times s_z\| dz d\theta$$
$$= \int_0^{2\pi} \int_0^{\frac{4}{3}} 3\sqrt{10} z dz d\theta$$
$$= \frac{7\sqrt{10}}{3}\pi$$

8. (5 points) *A torus of revolution* (doughnut) is obtained by rotating a circle C in the xz plane about the z-axis. Suppose that C has a radius r and center $(R, 0, 0)$.

   (a) Find a parametrization $\mathbf{r}(u, v)$ of the torus. Specify the set D in which (u, v) must lie.

   Hint: You can choose let u represent the angle that the line from the point $\mathbf{r}(u, v)$ on the torus to the center of the rotated circle form with the xy-plane, and let v denote the angle formed by the line from the point $\mathbf{r}(u, v)$ on the torus to the origin with the positive x-axis. See figures below.





Solution: First, consider the circle C in the xz-plane: $y = 0$ and $(x - R)^2 + z^2 = r^2$. $x - R = r\cos u$ and $z = r\sin u$, where $u \in [0, 2\pi]$. Then, the parametric equations for (x, z) on C is:

$$x = R + r\cos u, \quad z = r\sin u, \quad u \in [0, 2\pi].$$

We revolve the circle about the z-axis. Hence,

$$x = (R + r\cos u)\cos v, \quad y = (R + r\cos u)\sin v, \quad z = r\sin u, \quad u \in [0, 2\pi], \; v \in [0, 2\pi].$$

We can also write this as:

$$\mathbf{r}(u, v) = (R + r\cos u)\cos v \; \mathbf{i} + (R + r\cos u)\sin v \; \mathbf{j} + r\sin u \; \mathbf{k},$$

$$(u, v) \in D = [0, 2\pi] \times [0, 2\pi].$$

(b) Show that the surface area of the torus is $4\pi^2 Rr$.

Solution: We first find $|\mathbf{r}_u \times \mathbf{r}_v|$:

$$\begin{aligned}
\mathbf{r}_u(u, v) &= \langle -r\sin u\cos v, -r\sin u\sin v, r\cos u\rangle, \\
\mathbf{r}_v(u, v) &= \langle -(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0\rangle, \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle -r(R + r\cos u)\cos u\cos v, -r(R + r\cos u)\cos u\sin v, -r(R + r\cos u)\sin u\rangle.
\end{aligned}$$

Therefore,

$$|\mathbf{r}_u \times \mathbf{r}_v| = r(R + r\cos u).$$

Then, the surface area of the torus is

$$\begin{aligned}
\iint_D |\mathbf{r}_u \times \mathbf{r}_v| &= \int_0^{2\pi}\int_0^{2\pi} r(R + r\cos u)\; du\; dv \\
&= \int_0^{2\pi} 2\pi Rr + r^2\sin(2\pi) - r^2\sin(0)\; dv \\
&= 2\pi rR(2\pi) = 4\pi^2 Rr,
\end{aligned}$$

as desired.

9. (5 points) Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the surface S that is the portion of the surface $y^2 + 4z = 16$ that lies between the planes $x = 0$, $x = 1$, and $z = 0$.

Solution: Parametrize S by
$$s(x,y) = (x, y, 4 - \frac{1}{4}y^2)$$
where $0 \le x \le 1$ and $-4 \le y \le 4$. Then
$$\begin{aligned}
\int_S g d\Sigma &= \int_{-4}^{4}\int_0^1 g(s(x,y))\|s_x \times s_y\| dxdy \\
&= \int_{-4}^{4}\int_0^1 x\sqrt{y^2+4}\sqrt{1+\frac{1}{4}y^2} dxdy \\
&= \frac{1}{2}\int_{-4}^{4}\int_0^1 x(y^2+4) dxdy \\
&= \frac{56}{3}
\end{aligned}$$

10. (5 points) Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the vector field $\mathbf{F}(x,y,z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$, where S is the helicoid $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$, $0 \le u \le 1, 0 \le v \le \pi$, with upward orientation.

Solution: We have
$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\pi}\int_0^1 \mathbf{F}(r(u,v)) \cdot (r_u \times r_v) dudv & (1) \\
&= \int_0^{\pi}\int_0^1 (v\sin v - u\sin v\cos v + u^2\cos v) dudv & (2)
\end{aligned}$$
Since both $\cos v$ and $\cos v \sin v$ have integral 0 from 0 to 2π, the last one becomes,
$$(2) = \int_0^{\pi}\int_0^1 v\sin v \ dudv = \pi$$

11. (5 points) Find the outward flux of the field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the surface of the portion of the sphere $x^2 + y^2 + z^2 \le 25$ above the plane $z = 3$.

Solution: We first parametrize the surface S:
$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \sqrt{25 - x^2 - y^2}\mathbf{k},$$
where $(x,y) \in D = \{(x,y) \mid x^2 + y^2 \le 16\}$ (since we want $z \ge 3$, then $x^2 + y^2 \le 25 - 9$).
Hence,
$$\begin{aligned}
\mathbf{r}_x &= \langle 1, 0, -x/\sqrt{25 - x^2 - y^2}\rangle \\
\mathbf{r}_y &= \langle 0, 1, -y/\sqrt{25 - x^2 - y^2}\rangle \\
\mathbf{r}_x \times \mathbf{r}_y &= \langle x/\sqrt{25 - x^2 - y^2}, y/\sqrt{25 - x^2 - y^2}, 1\rangle.
\end{aligned}$$
Therefore,
$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = x^2 + y^2 + 1,$$

and

$$\begin{aligned}
\iint_S \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\ dS &= \iint_D x^2 + y^2 + 1\ dA \\
&= \int_0^{2\pi} \int_0^4 (r^2+1)r\ dr\ d\theta \\
&= 72(2\pi) = 144\pi.
\end{aligned}$$

The following problems are important, but they will not be graded. It is encouraged you work through them.

12. Use the surface integral in Stokes' Theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$ and C is thee intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$, counterclockwise when viewed from above.

> **Solution:** We have
> $$\text{curl}(\mathbf{F}) = (0, 0, -3x^2y^2)$$
> Parametrize the surface using spherical coordinate with $\rho = 4$.
> $$(x, y, z) \mapsto (4\cos\theta\sin\phi, 4\sin\theta\sin\phi, 4\cos\phi)$$
> where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{6}$.
> Calling this $\mathbf{S}(\theta, \phi)$, we have
> $$\begin{aligned}(\mathbf{S}_\theta, \mathbf{S}_\phi) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4\sin\theta\sin\phi & 4\cos\theta\sin\phi & 0 \\ 4\cos\theta\cos\phi & 4\sin\theta\cos\phi & -4\sin\phi \end{vmatrix} \\ &= (-16\cos\theta\sin^2\theta, -16\sin\theta\sin^2\phi, -16\sin\phi\cos\phi)\end{aligned}$$
> Thus by Stokes,
> $$\begin{aligned}\int_c \mathbf{F} \cdot d\mathbf{r} &= \iint_s (\text{curl}\mathbf{F}) \cdot d\mathbf{s} \\ &= \int_0^{2\pi}\int_0^{\pi/6} (0, 0, -768\cos^2\theta\sin^2\theta\sin^4\phi) \cdot (4\cos\theta\sin\phi, 4\sin\theta\sin\phi, 4\cos\phi)\ d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/6} -3072\cos^2\theta\sin^2\theta\sin^5\phi\ d\phi d\theta \\ &= -\frac{24}{5}\pi\end{aligned}$$

13. Let $\mathbf{n}$ be the outer unit normal of the surface S given by $4x^2 + 9y^2 + 36z^2 = 26, z \geq 0$, and let $\mathbf{F}(x, y, z) = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2}\sin(e^{\sqrt{xyz}})\mathbf{k}$.
Find the value of
$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}.$$
Hint: One parametrization of the ellipse at the base of the shell is of the form $x = a\cos(t), y = b\cos(t)$, for some constants a, b.

> **Solution:** For $-z = \theta$ the curve $4x^2 + 9y^2 = 26$ is parametrized by
> $$(x, y) \mapsto \left(\frac{\sqrt{26}}{2}\cos\theta, \frac{\sqrt{26}}{3}\sin\theta, \theta\right)$$

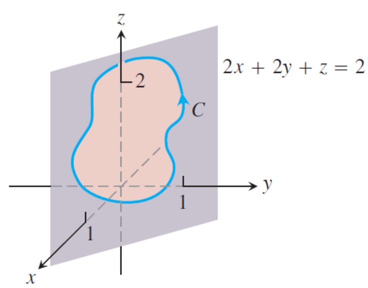
Thus we have

$$
\begin{aligned}
\iint_s \text{curl}\mathbf{F} \cdot d\mathbf{s} &= \int_c \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \left(\sqrt{26}3 \sin\theta, \frac{26}{4}\cos^2\theta, \left(\frac{26}{4}\cos^2\theta + \left(\frac{26}{9}\right)^2 \sin^4\theta \right)^{3/2} \sin() \right) \\
&\qquad \cdot \left(-\frac{\sqrt{26}}{2}\sin\theta, \frac{\sqrt{26}}{3}\cos\theta, 0 \right) d\theta \\
&= \int_0^{2\pi} \left(-\frac{26}{6}\sin^2\theta + \frac{26^{3/2}}{12}\cos^3\theta \right) d\theta \\
&= -\frac{13}{3}\pi
\end{aligned}
$$

14. **Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented as shown here. Show that

$$\int_C 2y\ dx + 3z\ dy - x\ dz$$

depends only on the area of the region enclosed by C and not on the position or shape of C.



Solution: For $\mathbf{F} = (2y, 3z, -x)$, we have curl$\mathbf{F} = (-3, 1, 2)$. Since C is on the plane $2x+2y+z = 2$, the surface enclosed by C is on the plane and everywhere has normal vector $(2, 2, 1)$, so the surface area is

$$d\mathbf{S} = \frac{(2,2,1)}{3} dA$$

Hence, by Stokes,

$$
\begin{aligned}
\int_C 2y\ dx + 3z\ dy - x\ dz = \int_C \mathbf{F} \cdot (dx, dy, dz) = \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (-3, 1, 2) \cdot \frac{(2,2,1)}{3} dA \\
&= -\frac{2}{3}(\text{Area of S})
\end{aligned}
$$

15. Use the divergence theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D, where $\mathbf{F}(x, y, z) = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$ and D is the region inside the solid cylinder $x^2 + y^2 \leq 4$, between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$.

Solution: We have $\nabla \cdot \mathbf{F} = (x-1) \mapsto (r\cos\theta - 1)$, so

$$\begin{aligned}
\iint \mathbf{F} \cdot d\mathbf{S} &= \iiint_D (\nabla \cdot \mathbf{F}) dV \\
&= \int_0^{2\pi} \int_0^4 \int_0^{r^2} (r\cos\theta - 1)r \; dzdrd\theta \\
&= \int_0^{2\pi} \int_0^4 \int_0^{r^2} (r^2\cos\theta - r) \; dzdrd\theta \\
&= -128\pi
\end{aligned}$$

16. **Among all rectangular solids defined by the inequalities $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq 1$, find the one for which the total flux of $\mathbf{F}(x,y,z) = (-x^2 - 4xy)\mathbf{i} - 6yz\mathbf{j} + 12z\mathbf{k}$ outward through the six sides is the greatest. What is the value of the greatest flux?

Solution: First we have $\nabla \cdot F = (-2x - 4y) + (-6z) + 12$, so

$$\begin{aligned}
\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^b \int_0^a (\nabla \cdot F) \; dxdydz \\
&= \int_0^1 \int_0^b \int_0^a (-2x - 4y - 6z + 12) \; dxdydz \\
&= -a^2b - 2ab^2 - 3ab + 12ab \\
&= -a^2b - 2ab^2 + 9ab
\end{aligned}$$

To maximize, we take $\nabla(-a^2b - 2ab^2 + 9ab) = (-2ab - 2b^2 + 9b, a^2 - 4ab + 9b) = 0$. Solving for a and b gives, for $a \geq 0$, $b \geq 0$,

$$\begin{aligned}
&\text{If } a = 0 \text{ and } b = 9/2, \text{ then }, -a^2b - 2ab^2 + 9ab = 0 \\
&\text{If } a = 3 \text{ and } b = 3/2, \text{ then }, -a^2b - 2ab^2 + 9ab = 27/2 \\
&\text{If } a = 9 \text{ and } b = 0, \text{ then }, -a^2b - 2ab^2 + 9ab = 0 \\
&\text{If } a = 0 \text{ and } b = 0, \text{ then }, -a^2b - 2ab^2 + 9ab = 0
\end{aligned}$$

So the maximum flux is $27/2$ and occurs for $a = 3, b = 3/2$.