Quiz #6 for Calculus 3 (MATH-UA.0123-001)

Problem 1. Let:

$$y_0(x) = (1-x)^2, \qquad y_1(x) = \cos(\pi x/2),$$

and let:

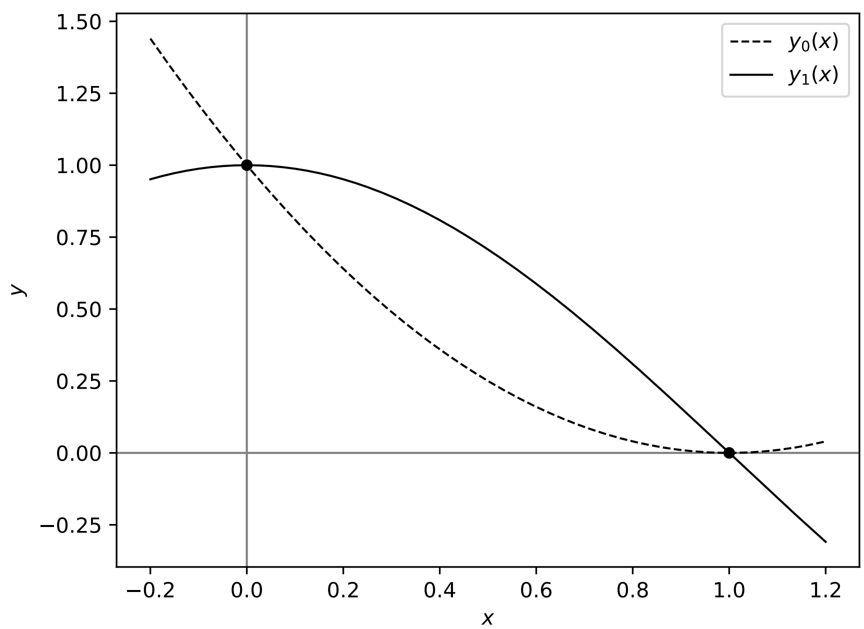
$$D = \Big\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } y_0(x) \leq y \leq y_1(x)\Big\}.$$

Evaluate the integral:

$$\iint_D (x + 2y) dA.$$

Hint: use integration by parts. [*10 points*]

This is an integral over a general region whose top and bottom boundaries are parametrized by the functions $y_0(x)$ and $y_1(x)$. This is what the region looks like:



The range of integration in the x variable is $0 \leq x \leq 1$. Note that if $x = 0$, then $y_0(0) = 1 = y_1(0)$, and if $x = 1$, then $y_0(1) = 0 = y_1(1)$.

The integral is:

$$\int_0^1 \int_{(1-x)^2}^{\cos(\pi x/2)} (x+2y)\,dy\,dx = \int_0^1 \left(xy + y^2\right)\Big|_{y=(1-x)^2}^{\cos(\pi x/2)} dx$$
$$= \int_0^1 \left(x\cos(\tfrac{\pi x}{2}) + \cos(\tfrac{\pi x}{2})^2 - x(1-x)^2 - (1-x)^4\right) dx.$$

Let's integrate the first term. Integrating by parts gives:

$$\int_0^1 x\cos(\tfrac{\pi x}{2})dx = \frac{2}{\pi}x\sin(\tfrac{\pi x}{2})\Big|_{x=0}^{1} - \frac{2}{\pi}\int_0^1 \sin(\tfrac{\pi x}{2})dx$$
$$= \frac{2}{\pi} - \frac{2}{\pi}\int_0^1 \sin(\tfrac{\pi x}{2})dx = \frac{2}{\pi}\left(1 + \frac{2}{\pi}\cos(\tfrac{\pi x}{2})\Big|_{x=0}^{1}\right)$$
$$= \frac{2}{\pi}\left(1 + \frac{2}{\pi}(0-1)\right) = \frac{2}{\pi}\cdot\frac{\pi-2}{\pi} = \frac{2(\pi-2)}{\pi^2}.$$

For the second term, we use the half-angle identity to get:

$$\int_0^1 \cos(\tfrac{\pi x}{2})dx = \int_0^1 \left(\frac{1+\cos(\pi x)}{2}\right)dx$$
$$= \frac{1}{2} + \frac{1}{2\pi}\sin(\pi x)\Big|_{x=0}^{1} = \frac{1}{2} + \frac{1}{2\pi}(0-0) = \frac{1}{2}.$$

For the last two terms, integrate:

$$\int_0^1 \left[x(1-x)^2 + (1-x)^4\right]dx = \int_0^1 \left[x - 2x^2 + x^3 + (1-x)^4\right]dx$$
$$= \frac{1}{2}x^2 - \frac{2}{x}^{3} + \frac{1}{4}x^4 - \frac{1}{5}(1-x)^5\Big|_{x=0}^{1}$$
$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} = \frac{17}{60}.$$

Putting these together gives:

$$\int_0^1 \int_{(1-x)^2}^{\cos(\pi x/2)} (x+2y)\,dy\,dx = \frac{2(\pi-2)}{\pi^2} + \frac{1}{2} - \frac{17}{60} = \frac{2(\pi-2)}{\pi^2} + \frac{13}{60} = 0.448\ldots$$