Quiz #8 for Calculus 3 (MATH-UA.0123-001)

Problem 1. A solid E lies within the cylinder $x^2 + y^2 = 4$, below the plane $z = 4$, and above the paraboloid $4 - x^2 - y^2$. The density (units: kg/m^3) at any point is equal to C times the distance to the z axis. Find the mass of E (in kg). [*5 points*]

This integral is easiest to do in cylindrical coordinates. Since $r^2 = x^2 + y^2$, we have:

$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_{4-r^2}^4 Cr^2 dz dr d\theta &= 2\pi C\int_0^2 r^2\int_{4-r^2}^4 dz dr \\ &= 2\pi C\int_0^2 r^2\Big[4-(4-r^2)\Big]dr \\ &= 2\pi C\int_0^2 r^4 dr = 2\pi C\,\frac{1}{5}r^5\Big|_{r=0}^2 = \frac{64}{5}\pi C.\end{aligned}$$

Problem 2. Evaluate $\iiint_E (x^2 + y^2) dV$, where E is the region bounded by the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$. Hint: $\sin(\phi)^3 = \frac{1}{4}(3\sin(\phi) - \sin(3\phi))$. [*5 points*]

Since we're integrating over the spherical shell with inner radius $r_0 = 2$ and outer radius $r_1 = 3$, we should use spherical coordinates. To rewrite the integrand using spherical coordinates, we can simply substitute:

$$x^2+y^2 = \rho^2\cos(\theta)^2\sin(\phi)^2+\rho^2\sin(\theta)^2\sin(\phi)^2 = \rho^2\Big[\cos(\theta)^2+\sin(\theta)^2\Big]\sin(\phi)^2 = \rho^2\sin(\phi)^2.$$

Alternatively, we can notice that:

$$x^2 + y^2 = r^2 = (\rho\sin(\phi))^2 = \rho^2\sin(\phi)^2.$$

We integrate:

$$\int_0^{2\pi}\int_0^{\pi}\int_2^3 = \rho^4\sin(\phi)^3 d\rho d\phi d\theta = 2\pi\int_0^{\pi}\sin(\phi)^3 d\phi\int_2^3\rho^4 d\rho.$$

Using the hint, we can compute $\int_0^{\pi}\sin(\phi)^3 d\phi = 4/3$. At the same time, $\int_2^3\rho^4 d\rho = 211/5$. Hence, the integral is:

$$\iiint_E (x^2 + y^2) dV = 2\pi\cdot\frac{4}{3}\cdot\frac{211}{5} = \frac{1688}{15}\pi = 353.533\ldots$$