Quiz #9 for Calculus 3 (MATH-UA.0123-001)

Problem 1. Let $\boldsymbol{F}(x, y) = (3 + 2xy)\boldsymbol{i} + (x^2 - 3y^2)\boldsymbol{j}$. Find a function f such that $\boldsymbol{F} = \nabla f$. Be careful of any constants of integration. [*3 points*]

> Let $f(x, y) = 3x + x^2y - y^3 + C$, where C is a constant. Then:
>
> $$f_x(x, y) = 3 + 2xy, \qquad f_y(x, y) = x^2 - 3y^2,$$
>
> from which we can conclude $\boldsymbol{F}(x, y) = \nabla f(x, y)$. Note that there are infinitely many different choices of f, each corresponding to a different constant $C \in \mathbb{R}$.

Problem 2. For the same $\boldsymbol{F}$ as in Problem 1, evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is the curve given by $\boldsymbol{r}(t) = e^t \sin(t)\boldsymbol{i} + e^t \cos(t)\boldsymbol{j}$, for t such that $0 \leq t \leq \pi$. [*3 points*]

> Since $\boldsymbol{F}$ is a conservative (or "gradient") vector field, we can compute the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ by taking the difference between f evaluated at the start and end of the curve C. Since C is parametrized from $t = 0$ to $t = \pi$, and its endpoints are:
>
> $$\boldsymbol{r}(0) = (0, 1), \qquad \boldsymbol{r}(\pi) = (0, -e^{\pi}),$$
>
> we have:
>
> $$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(0, 1) - f(0, -e^{\pi}) = \big(e^{3\pi} + C\big) - \big((-1)^3 + C\big) = e^{3\pi} + 1.$$

Problem 3. Let C be the circle with radius 2 centered at the origin. Evaluate the line integral $\oint_C (x - y)dx + (x + y)dy$ directly and using Green's theorem. [*2 points*]

> Let D denote the disk of radius 2 centered at the origin, so that $\partial D = C$. Then, if we let $P = x - y$ and $Q = x + y$ (note that $P_y = -1$ and $Q_x = 1$), using Green's theorem we can write:
>
> $$\begin{aligned} \oint_C (x - y)dx + (x + y)dy &= \oint_{\partial D} P dx + Q dy = \iint_D (Q_x - P_y) dA \\ &= \iint_D (1 - (-1)) dA = 2 \iint_D dA = 8\pi. \end{aligned}$$

The last equality follows by observing that $\iint_D dA$ is the area of the disk of radius 2, which is $\pi r^2 = 4\pi$, where $r = 2$.

To compute the line integral directly, we need to parametrize C and do the line integral. We can parametrize C by writing:

$$\boldsymbol{r}(t) = 2(\cos(t), \sin(t)), \qquad \boldsymbol{r}'(t) = 2(-\sin(t), \cos(t)).$$

If we let $\boldsymbol{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$, then:

$$\begin{aligned}
\oint_C P dx + Q dy &= \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt \\
&= 4 \int_0^{2\pi} (\cos(t) - \sin(t), \cos(t) + \sin(t)) \cdot (-\sin(t), \cos(t))\, dt \\
&= 4 \int_0^{2\pi} \Big(-\sin(t)\cos(t) + \sin(t)^2 + \cos(t)^2 + \sin(t)\cos(t) \Big) dt \\
&= 4 \int_0^{2\pi} dt = 4 \cdot 2\pi = 8\pi.
\end{aligned}$$

By comparing the two approaches you can see that Green's theorem lets us compute this integral much more simply and with a lower likelihood of making mistakes.