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MATH-UA 252-005 / MA-UY.4424-C - Midterm #1

1. Newton's method and minimization:

1. Show how to use Newton's method to minimize a function $f(x)$.
2. Prove that this is equivalent to minimizing a sequence of quadratic polynomials. (*Hint:* use a Taylor expansion to find the quadratic polynomial to be minimized.)

1. Let x^* be minimizing argument of f.

we require $f'(x^*) = 0$. ("first-order necessary conditions for optimality")

2. $$f(x_n + \Delta x) = f(x_n) + f'(x_n)\Delta x + \frac{1}{2} f''(x_n)\Delta x^2 + O(\Delta x^3)$$

$$\underset{\text{define}}{=} p_n(\Delta x) + O(\Delta x^3).$$

$$p_n'(\Delta x) = f'(x_n) + f''(x_n)\Delta x = 0$$

$$\Rightarrow \Delta x = \frac{-f'(x_n)}{f''(x_n)}.$$

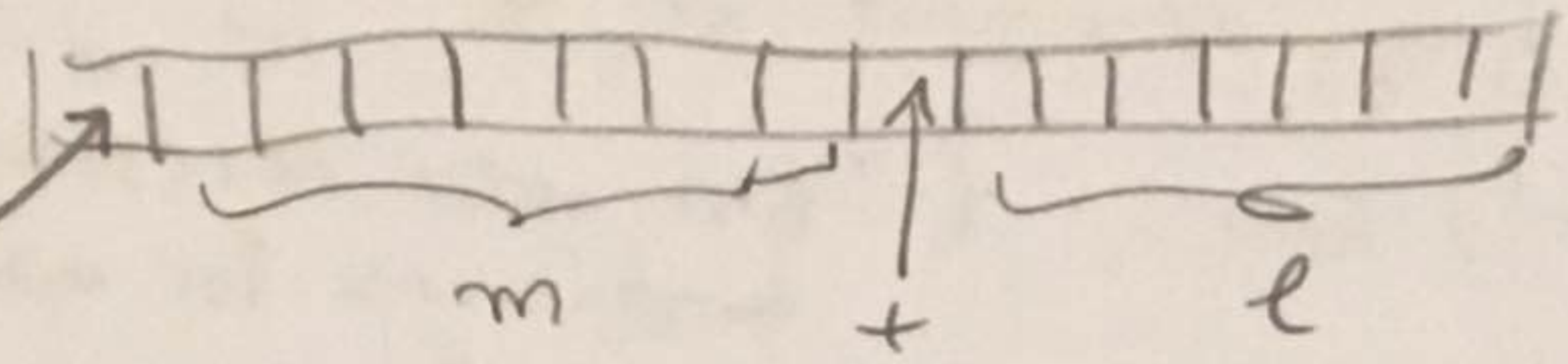
If we set $x_{n+1} = x_n + \Delta x = x_n - \frac{f'(x_n)}{f''(x_n)}$,

we just get the Newton iteration for solving $f'(x) = 0$.

2. Floating-point numbers:

1. Make up and explain a simple floating-point representation which uses 16 bits.
2. Let $x \in [0, 1]$. We can write $x = t_{-1}3^{-1} + t_{-2}3^{-2} + t_{-3}3^{-3} + \cdots$ for some coefficients t_k, where $t_k \in \{0, 1, 2\}$. We call each coefficient t_k a *trit*. Find this expansion for $x = 1/10$. (*Hint:* each t_k must be nonnegative!)

1. $\pm m \times 2^{\pm e}$



2. $\frac{1}{10} = \frac{1}{1+3^2} = \frac{1}{3^2} \cdot \frac{1}{1+3^{-2}}$

geometric series $= \frac{1}{3^2}\left(3^0 - 3^{-2} + 3^{-4} - 3^{-6} + 3^{-8} - 3^{-10} + \cdots\right)$

$3^0 - 3^{-2} = 1 - \frac{1}{9} = \frac{10}{9} = 3^{-2}(2 \cdot 3^1 + 2 \cdot 3^0)$
$= 2 \cdot 3^{-1} + 2 \cdot 3^{-2}$

$= 3^{-2} \cdot \left(2 \cdot 3^{-1} + 2 \cdot 3^{-2} + 2 \cdot 3^{-5} + 2 \cdot 3^{-6} + \cdots\right)$

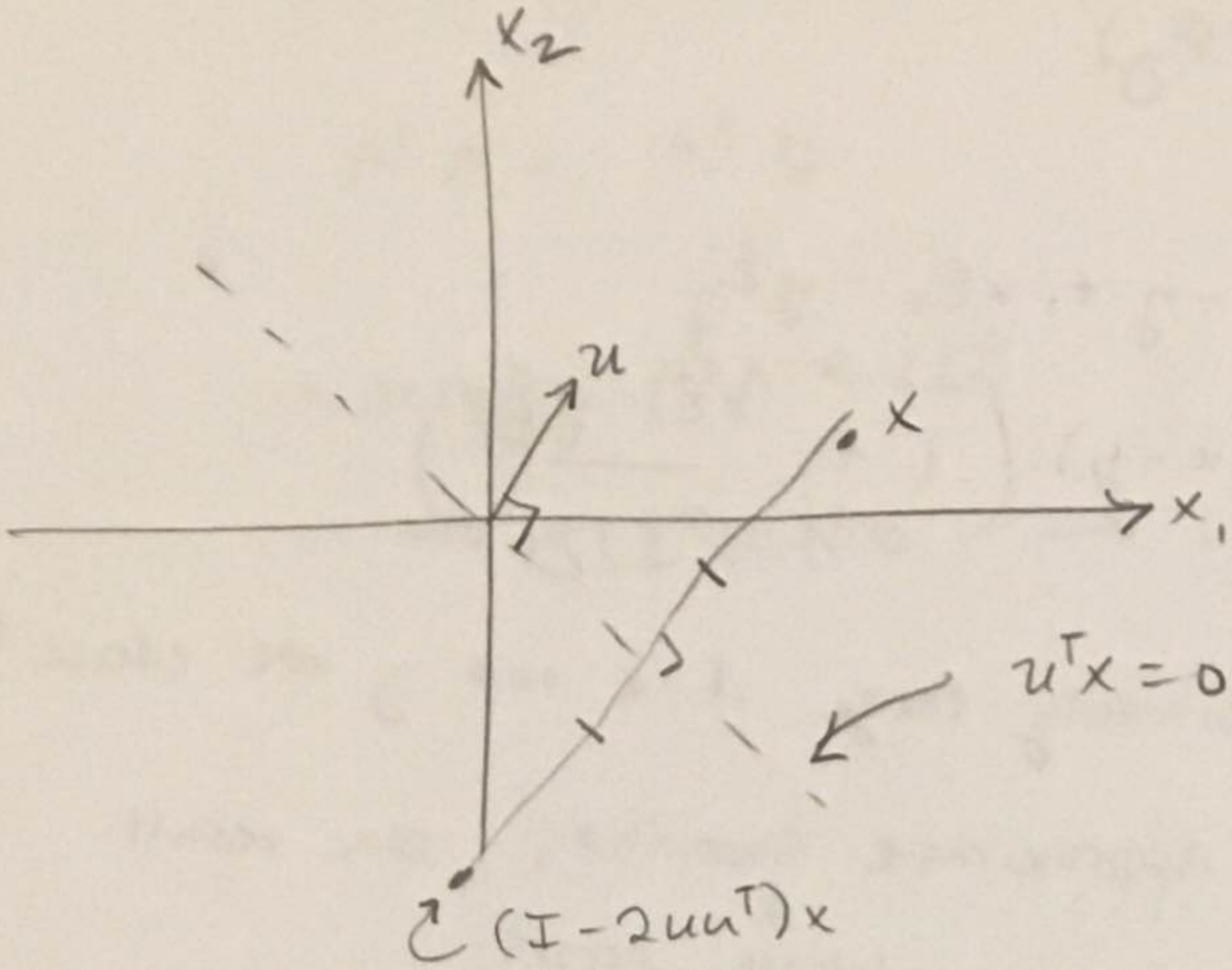
$= 0.00220022002200\ldots_3$

3. Linear algebra.

1. Let $u \in \mathbb{R}^n$ be a unit vector, and let $R_n = I - 2uu^T$. Draw a picture for $n = 2$ which shows what multiplying with R_n does. For general n, compute the eigenvalues of R_n (including their multiplicity), explain what the eigenspaces are (the subspaces of R^n corresponding to each distinct eigenvalue), and compute $\det(R_n)$.

2. Prove that if A is positive definite, then $a_{ii} > 0$ for each i.

1.



2. $a_{ii} = e_i^T A e_i > 0$.

Eigenvalues: two cases:

i.) $u^T x = 0 \Rightarrow (I - 2uu^T)x = x$

$\Rightarrow$ eigenvalue $= 1$

ii) $u^T x \neq 0$: let $x = u$

$\Rightarrow (I - 2uu^T)u = u - 2|u|^2 u$

$= (1 - 2|u|^2)u$

$\Rightarrow$ eigenvalue $= 1 - 2|u|^2$

The first eigenvalue has multiplicity $n-1$, and the second has multiplicity 1, since the dimension of the plane $u^T x = 0$ is $n-1$.
This describes the two eigenspaces.
Now we can conclude that $\det(I - 2uu^T) = 1 - 2|u|^2$.

4. Sources of error. Compute the relative error of subtraction and explain the phenomenon of *catastrophic cancellation.*

$$\hat{x} = x(1+\varepsilon_x)$$

$$\hat{y} = y(1+\varepsilon_y)$$

$$\hat{x} - \hat{y} = x - y + x\varepsilon_x - y\varepsilon_y$$

$$= (x-y)\left(1 + \frac{x\varepsilon_x - y\varepsilon_y}{x-y}\right).$$

This can be arbitrarily large if x and y are close!

So subtracting approximate quantities can result in arbitrarily large relative errors.

This is "catastrophic cancellation."

5. Numerical linear algebra. Let $Ax = b$ be an *overdetermined* linear system. Give a quick explanation of how to use the Cholesky decomposition to solve this overdetermined system. What must be true of A for this to work? Why?

Normal equations:

$$A^T A x = A^T b$$

cholesky: $A^T A = LL^T$

$$\Rightarrow \quad LL^T x = A^T b \quad \rightarrow \quad L^T x = \underbrace{L^{-T} \underbrace{L^{-1} \underbrace{A^T b}_{\text{mult } (O(mn))}}_{\text{tri solve } (O(n^2))}}_{\text{tri solve } (O(n^2))}$$

Cholesky decomposition exists for spd matrices.

1) $A^T A$ is symmetric ($\underline{s}$pd) ✓

2) if A has full column rank, then $A^T A$ is positive definite (s$\underline{pd}$) ✓

Why? If A is not full column rank, must be a vector u such that $Au = 0$. Hence $A^T A u = 0$. So $A^T A$ has at least one zero eigenvalue. So $A^T A$ cannot be pd.

But observe that $z^T A^T A z = \|Az\|^2 \geq 0$. for any $z \neq 0$. So $A^T A$ is at least positive semi def.

Hence, full column rank $\Rightarrow$ $A^T A$ pos def.

6. The Babylonian algorithm. The iteration for the Bablyonian algorithm is $x_{n+1} \leftarrow (y/x_n + x_n)/2$, where $y = x^2$. We saw that if $y > 0$ and $x_0 > 0$, then $x_n \to \sqrt{y} = x$. Consider the error $e_n = x_n - x$. The sequence $x_n \to x$ with an order of convergence equal to $q > 0$ if:

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^q} = \mu \in [0, 1]. \qquad (1)$$

If we pick $q = 1$ and find that the limit above equals $\mu = 0$, then the sequence converges *superlinearly*. Assume that $x_n \to \sqrt{y} = x$ and show that the Babylonian algorithm converges superlinearly.

Note: $$\frac{1}{2}\left(\frac{x^2}{x_n} + x_n\right) - x = \frac{1}{2}\left(\frac{x^2}{x_n} + \frac{x_n^2}{x_n} - \frac{2xx_n}{x_n}\right)$$

$$= \frac{1}{2}\left(\frac{(x - x_n)^2}{x_n}\right)$$

So: $$\frac{e_{n+1}}{e_n} = \frac{1}{2}\frac{(x_n - x)^2}{x_n} \cdot \frac{1}{x_n - x} = \frac{1}{2}\frac{x_n - x}{x_n}$$

$$= \frac{1}{2}\left(1 - \frac{x}{x_n}\right)$$

Hence:

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{x}{x_n}\right)$$

$$= \frac{1}{2}\left(1 - \frac{x}{x}\right) = 0,$$

since $x_n \to x$ as $n \to \infty$!