Numerical Analysis Notes 01/31/2022

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1 Announcements

<u>Office Hours:</u>
Tuesday: 7:00pm-9:00pm Via Zoom
Wednesday: 5:00pm-6:00pm, 2MTC room 872 with Professor Potter
Friday: 1:00pm-2:00pm, 2MTC room 858 with Mariana
Class time: 3:30pm-4:50pm

2 Note on Programming Homework 1

Brentq: A 'hybrid rootfinder' which implements Brent's method.
Brentq solves: "find $x \in [a, b]$ such that $f(x) = 0$"
Signature: $x = brentq(f, a, b, tol = None)$

- tol: "tolerance," how accurately the root is found.
- f: An instance of something on Python which is callable. . . so if I write
- "$f(x)$", where x is a float, then $f(x)$ is a float
  From scipy.optimize import brentq
  From Ipython type ?brentq for docs or just google "brentq scipy"

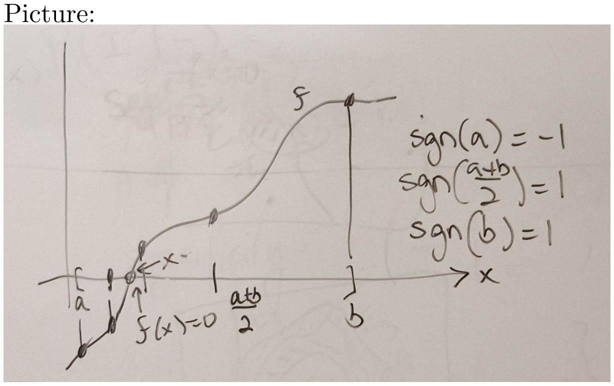
Question: $p(x) = (x - 10^{-16})(x + 10^{-16}) = x^2 - 10^{-32} \approx x^2$
Question: how accurate does findroots need to be?
Answer: roots = findroots(p,a,b,tol) (where tol > 0)

Def: an open set E in R is a subset of the real line such that if x $\in E$,then there exists some $\epsilon > 0$ such that $B_\epsilon = \xi y \in R : |x - y| < \epsilon$
Def: a set F $\subseteq R$ is closed if it's complement is open.
F^c ={y=R: y $\notin F$}
e.g. Is the set $[1, \infty)$ closed?
well $[1, \infty)^c = (-\infty, 1)$...*open*
$\Rightarrow [1, \infty)$ is closed.
Note: For the contraction mapping theorem, the assumption on the interval [a,b] is that it is closed AND bounded or compact

3 Iterative Solution of Equations

(or in 1D . . . "rootfinding".)

We want to solve $f(x) = 0$ for $f : R \to R$.
Let's just assume f $\in$C^0($[a, b]$) (this is closed and bounded)
Picture:



Last class : IVT (Intermediate Value Theorem)
$f \in C^0([a, b]), f(a) < 0, f(b) > 0$, then $\exists x \in E$ such that f(x) = 0.

Def: signum e.g. np.sign,

- sgn(x) = 1 if $x > 0$
- sgn(x) = 0 if x = 0
- sgn(x) = -1 if $x < 0$

3.1 Bisection

$a_0, b_0 = a, b$
k=0
while True:
    sign_left = np.sign(a_k)
    sign_mid = np.sign($\frac{a_k+b_k}{2}$)
    sign_right = np.sign(b_k)
    #check if any of the signs ==0
    if sign_left != sign_mid:
        a_{k+1}, $b_{k+1} = a_k$, $\frac{a_k+b_k}{2}$
        continue
    sign_mid != sign_right:
        a_{k+1}, $b_{k+1} = \frac{a_k+b_k}{2}$, b_k
        continue

3.2 Time Complexity (How fast is this?)

$\frac{|b_{k+1}-a_{k+1}|}{|b_k-a_k} = \frac{1}{2} \Rightarrow |b_k - a_k| = \frac{b-a}{2^k}$

Accuracy:
for rootfinding (solving $f(x) = 0$) there are 2 options:

1. $x - x_k$
2. $f(x) - f(x_k)$ ('residual')

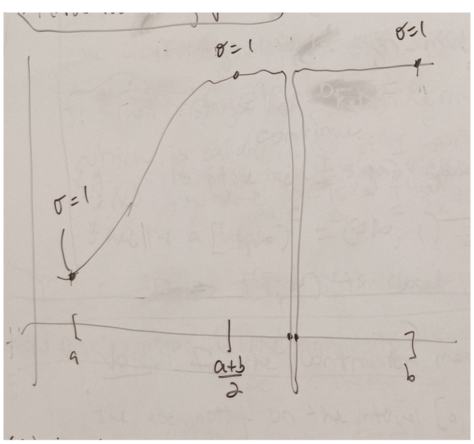
Question: how many steps to get $|x - x_k| < \epsilon$?
$\Leftrightarrow$ how big does k need to be to get $|b_k - a_k| < \epsilon$
$\epsilon =$—b$_k - a_k| = \frac{|b-a|}{2^k} \Rightarrow 2^k\epsilon = |b-a|$
$\Rightarrow$log$\epsilon + klog2 = log|b-a|$
k = $\frac{log\frac{1}{\epsilon}+log|b-a|}{log_2} = log_2\frac{1}{\epsilon} + log_2|b-a|$
k = $0(log\frac{1}{\epsilon})$
$\frac{log10^{10}}{log2} \approx 33$
What should we assume about f?
1) Completely problem dependent
2) Easy Rootfinding problem

- Exercise: 1) how to solve p(x) = 0, if p(x) = $ax^2 + bx + c$
- Answer: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Let's say f(x) = p(x) + ϵ(x)
What can we say about solving f(x) = 0?

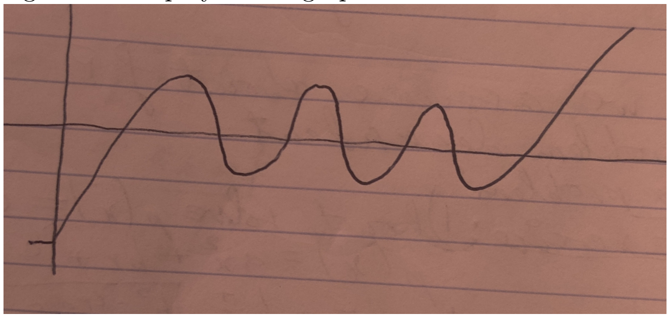
Professor's point: if ϵ(x) is small . . . then no need for bisection algorithm . . . just use the quadratic formula.

3.3 Hard rootfinding problem



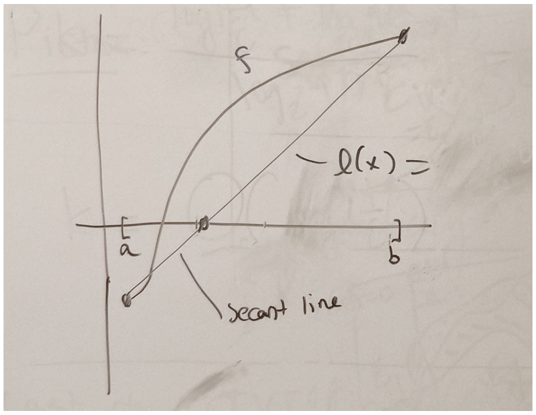
Also look at Weierstrass Monster function and the solution of a stochastic dif-

ferential equation ('stock price graph')
e.g. This is a polynomial graph:



Then: Sturm's theorem gives us global information to guide our search . . .

How do we make bisection go faster . . . assuming we have a reasonable problem to solve?

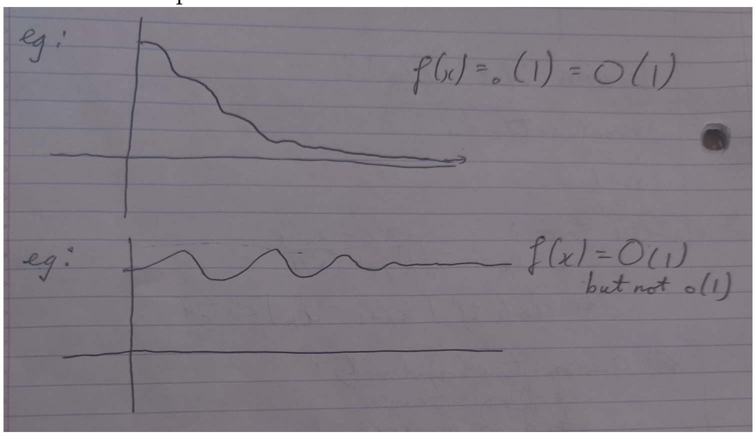


Exercise: what is l(x)?// Secant method:// $x_0 = a$
$x_1 = b$
$k = 2$
while true:
$x_k = x_{k-1} - f_k(\frac{x_k - x_{k-1}}{f_k - f_{k-1}})$
check if $|x_{k+1} - x_k| < tol$
Question: time complexity?

- rate of convergence
- orders of convergence

Landau notation: big O notation . . .
Def: f(x) = O(g(x)) [as x → x]
if :$\lim_{x_0} \frac{|f(x)|}{|g(x)|} = 0$

Some examples:



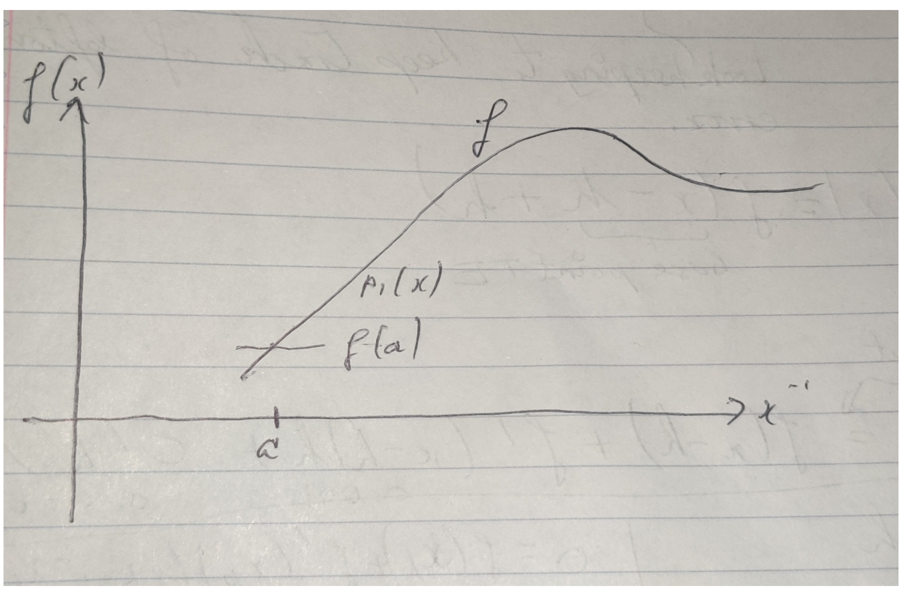
Def: let f: $R \to R$ be a C^{∞} (infinitely differentiable function) in a ball surrounding a point $a \in R$. Then there exists a polynomial (called the Taylor polynomial of order k).

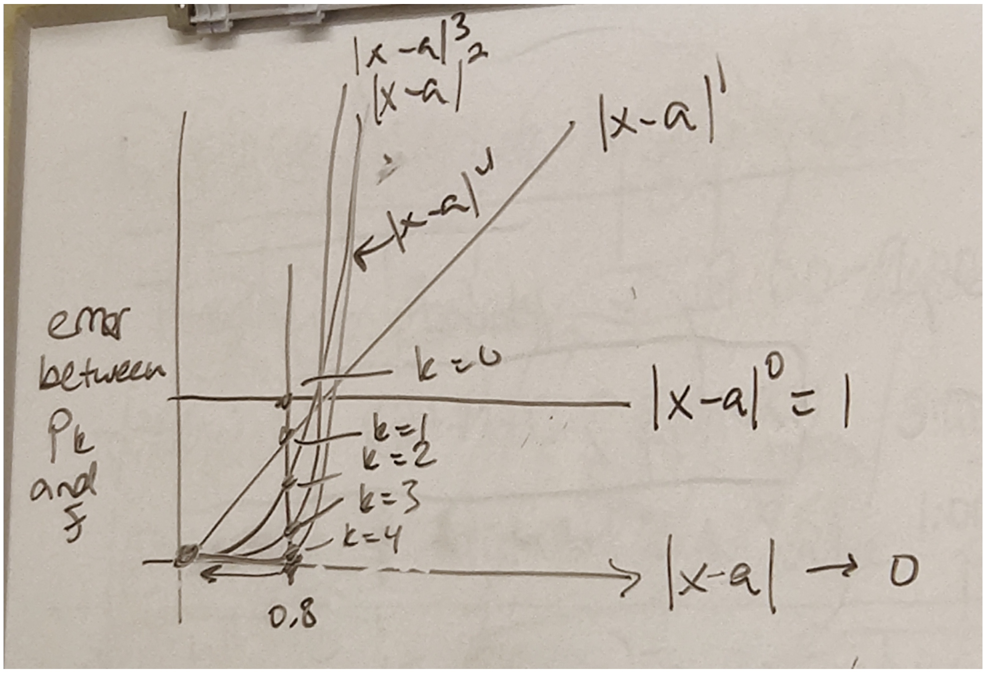
$p_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}}{3!}(x-a)^3 + ... + \frac{f^{(k)}}{k!}(x-a)^k$

$= \sum_{m=0}^{k} \frac{f^{(m)}}{m!}(x-a)^m$

Such that: the remainder

$$R_k(x) = f(x) - p_k(x) = O(|x-a|^k) \qquad \text{as x} \to a$$





More useful forms of Taylor Expansions (TEs) for our purposes:
f(x + h) = f(x) + f'(x)h + f''(x)$\frac{h^2}{2} + O(h^3)$ [or $o(h^2)$]

- $O(h^3)$ is for bookkeeping to keep track of leftover error

O = f(x) = f(x - h + h)

- x - h is the base point TE

linearizing f about x - h = f(x - h) + f'(x - h)h + O(h^2)
Relabel: $x_k = x - h$
$x_k + 1 - x_k = x - x + h - h$

- $\triangle$x$_k$ is used as 'the step'

O = f(x_k) + f'(x_k)($x_{k+1} - x_k + (x_{k+1} - x_k)^2$ + O—$x_{k+1} - x_k|^2$
Rearrange:
$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} + O|x_{k+1} - x_k|^2$

1. Started with the thing we are solving
2. Taylor expanded it in a sensible way
3. Converted it into an iteration
4. Therefore we Taylor Expanded, we have an estimate of the error (more or less).

Exercise: In the derivation of Newton's method what happens if f'(x_k) $\sim$x$_{k+1} - x_k$